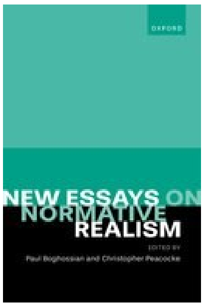

New Essays on Normative Realism
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CHAPTER

19 Reflections on Wittgenstein on the Normativity of Arithmetic

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Abstract

Elementary arithmetical truths are normally taken to be justifiable a priori if any truths are. They are also normally taken to be both necessary and empirically substantial—essentially applicable to our dealings with the world and empirically predictive, yet at the same time good for counterfactual reasoning about far-fetched and exotic scenarios, however extreme. This imbues them with a strong form of normativity: When results don't 'add up', the explanation, we are completely confident, will always be that some *outré* empirical factor is responsible. Simple arithmetic is inviolable. I will argue that scrutiny of the various methods—simple informal cognitive routines involving counting and pictures—whereby basic arithmetical judgements are initially apt to win our confidence serves to make it puzzling how they can *justify* this exalted status: how such procedures can entitle us either to the very high levels of confidence we standardly place in the judgements they lead to, or to the modal (counterfactual) significance we standardly attach to those judgements. The chapter will elaborate this apparent shortfall and review various proposals for redressing it, arguing that each comes short. A sceptical challenge will accordingly stand unanswered. The chapter will conclude by effecting a connection with the deflationary attitude to logic and mathematics that Wittgenstein canvassed in his *Remarks on the Foundations of Mathematics*.

Keywords: arithmetic, necessity, a priority, Wittgenstein, proof by pictures, proof by process
Subject: Epistemology, Moral Philosophy, Philosophy of Mathematics and Logic
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1 Introduction

Towards the end of the *Remarks on the Foundations of Mathematics*[1]—and, I am tempted to say, as if by way of a conclusion to all his thought about the status of mathematics—Wittgenstein wrote:

> What I am saying comes to this, that mathematics is *normative*. (1981: VII §61)

And a few lines later:

> Mathematics forms a network of norms. (1981: VII §67)

Readers familiar with the principal ideas in Wittgenstein's later writings about logic and mathematics will know better than to understand him as avowing merely that mathematics has 'some kind of sort-of-normative aspect'. Rather, his thought is that mathematics is *essentially normative*, where this involves, in Wittgenstein's view, a fundamental contrast with the idea that the business of mathematics is to chart mathematical reality, to uncover mathematical truths—the latter being an idea whose hold on our thinking about mathematics he regards as pernicious and fundamentally misconceived. For Wittgenstein, the normativity of mathematics wars with any kind of platonist (realist) philosophy of pure mathematics, the temptation to which is nourished, in his opinion, exactly by oversight of its essentially normative (regulative) character.

Reflecting on this aspect of Wittgenstein's thought, one cannot but be struck by its distance from the assumptions of contemporary philosophers who take the idea of normative realism seriously and accordingly reject the notion that there is any immediate tension, of the kind seemingly basic to Wittgenstein's thinking,
p. 456 between ↳ normativity and *representation*. It is a given of the way many, perhaps a majority of philosophers now conceive matters that, for at least some areas of normative thought, there simply is no such essential tension harboured by the very idea of normative *realism*: no solecism whatever in the idea of a judgement's being at once both normative and true on a realist understanding of truth.

'A realist sense understanding of truth?' There was a period in 20th-century philosophy when granting the aptitude for truth of a targeted class of judgements was thought be the exclusive prerogative of realism about their subject matter, and when various kinds of anti-realism about e.g ethics were accordingly at pains to argue that, the 'propositional surface' of ethical discourse notwithstanding, the true semantic function of its signature claims was other than to represent ethical truths—to argue that there are no ethical truths, strictly and soberly so described. However, in contrast with those now generally discarded forms of non-factualism and error-theory that were prevalent in his time, Wittgenstein himself would have had no quarrel with the idea that suitable statements of mathematics, ethics, aesthetics, and others are perfectly properly predicable by 'true'.[2] The misunderstanding of the status of mathematical judgements that he sought to expose was not the notion that suitable such judgements are perfectly unexceptionably characterisable by 'true'.

So what is the issue? I have myself argued at some length in other work that there are various separable strands in our lay-philosophical thinking about realism and objectivity.[3] Here, though, we can simplify to some extent. For, taking moral truth as a paradigm of normative truth if anything is, my impression is that the key issue in the recent and contemporary debates about moral realism concerns the *mind-independence* of moral truth—that is, the idea that when moral judgements are true, they are so in virtue of mind-independent facts.

This will be our focus for the discussion to follow. It is the idea of mind-independent but normative truth—the idea of normative facts that, in some important sense, are in no way 'of our making' and to which our thought can merely be responsive—that is of interest here. I will focus on elementary arithmetic as a case study for the very idea of normative realism as intuitively so characterised. The first task, however, will be to offer some clarification of the respects in which this basement-level fragment of pure mathematics is indeed properly regarded as normative—indeed more than that, as presenting as impressive a candidature for normative realism as we are likely to find. The main business of the chapter will then be to develop and review a plausibly Wittgensteinian line of argument which not merely presents a severe challenge to a normative realist view of
p. 457 elementary ↳ arithmetic, but which illustrates a form of challenge which confronts realism about any domain of normative thought.[4]

2 The Normativity of Elementary Arithmetic

Before going any further, we need to ask: was Wittgenstein right? Is mathematics—in particular is elementary arithmetic—normative? Is, say, 65+37=102 a normative judgment? What does it mean to say so?

We need, in talking of normative domains of thought, to set aside straightaway what are unquestionably purely descriptive *meta-level* judgements about the norms that actually operate in the given area: the kind of judgements that an anthropologist might make in describing the ethics of a primitive tribe, or such as one might find in a manual explaining the rules of chess. Rather, our concern is with a type of judgment an acceptance of which involves subjecting oneself to a state of governance by—or at least some form of responsiveness to or respect for—a practical or intellectual standard of some kind.

The most natural approach, of course, is to look for the essential marks of the normativity of a judgement in terms of its ingredient conceptual vocabulary. Normative judgements, surely, are those which entail something about what there is reason to do or think, or what one ought to do or think, or what is in one way or another (dis)valuable, commendable, or blameworthy.[5] Such an approach conceives the normativity of a judgment as a property of the kind of *content* it carries, a property conferred by that content's explicitly configuring one or another of a family of pre-selected normative concepts or as on analysis entailing a judgement explicitly involving such a concept or concepts. The resulting somewhat untidy landscape of proposals does, no doubt, gesture at a loose if legitimate sense of 'normative judgement'—a judgement that entails an evaluation, or something about what there is reason to do. I suggest however that we should hesitate to deem this general kind of proposal to be indicative of all cases worth the label 'normative p. 458 ↳ judgement' if only because, —if we do, —we will be obliged to rule Wittgenstein's idea of the normativity of mathematics out of court without so much as a hearing. There is no analytical entailment of any claim configuring concepts of these various kinds by 65+37=102 nor by any other mathematical judgement. If Wittgenstein's idea is not to be ruled out so simply, we need a conception of what makes for a normative judgement that broadens the approach.

What might be such a conception? It's familiar that rules and commands can be expressed in the indicative without containing any explicit marker of evaluation or reason-giving. For example, 'Lights will be out at 22.00', posted on the inside of the barracks door, or ' The Queen moves any number of unobstructed squares along a rank, file, or diagonal on which she sits' stated in the introductory chess manual. Yet an agent's acceptance of a rule or command, however expressed—including as a flat indicative as in these cases—is surely normative over their potential actions. Normativity, understood in the quite natural way illustrated by these examples, is not always a product of literal content, but sometimes is a matter, broadly, of the attitude either correctly or optionally taken to a content by one who produces or accepts it. The normativity of the lights-out command is grounded in the intention of the issuing authority, together with the recognition of that authority by the audience it is directed at. The normativity of the description of the Queen's move is grounded in the understanding that this is one aspect of how chess is correctly played.

It's notable that in *RFM* Wittgenstein is repeatedly drawn to the idea of assimilating the role of mathematical propositions to that of rules and/or commands.[6] Still, the idea is merely suggestive. What does 65+37=102 regulate? What is it a 'command' to do? In many places, Wittgenstein seems to be thinking in terms of the 'rules' of mathematics as regulative over mathematical practice itself—as determining what counts as correct calculation, for example. So conceived, 65+37=102 would be viewed as a rule partially determining what is to count as adding correctly. For our present purposes, however, it will be more fruitful to scout a different conception of mathematical normativity (albeit one that will not generalize as widely as is suggested by the two quotations from *RFM*, section VII with which we started)

We can approach the matter by returning to an idea mentioned above that is prominent in the discussions of moral realism: that of *mind-independence*. Here is a characteristic passage:

> A growing number of philosophers are sympathetic to moral realism (sometimes called robust moral
p. 459 realism). This is the view, roughly, that there are moral facts, ↳ and these facts are robustly mind-independent, in the sense that they are not constituted by people's attitudes, say, of approval and disapproval, nor are they constituted by the attitudes of hypothetical ideal observers. The wrongness of slavery, for example, according to moral realism, does not consist in the fact that anybody disapproves of it, nor does it consist in the fact that maximally rational, well-informed observers would disapprove of it. According to moral realism, actual and ideal observers alike play the role, at best, of trackers of moral facts, not determiners of them. So slavery would be wrong even if actual people approved of it, and even if ideal observers (specified in non-moral terms) approved of it.[7]

I'll return at the end to this particular claim about ethics—in intention, at least, that fundamental ethical truths are metaphysically independent of contingencies of our actual moral judgements and sensibilities. The point of immediate concern for our present dialectic is that the same is true in spades, so we think, of elementary arithmetic. 7+5=12, we are absolutely confident, would have been true even if no one had ever done any arithmetic, or had any arithmetical thoughts, or had had anything but what we would regard as crazily misguided arithmetical thoughts. It would be true in any circumstances whatever. It is, we think, an *absolute* truth—independent of any contingencies, not just contingencies of judgement—and we treat it accordingly.

Care is needed here. The important point is not, or not merely, that the truths of elementary arithmetic are necessarily true, and thus true independently of anyone's actual propensities of judgement, but concerns our *attitude* to arithmetic. The notion of necessary truth is after all a philosophical sophistication of something more basic. That more basic thing—a concept with which philosophically unsophisticated thinkers who lack explicit concepts of modality can nevertheless be credited—is what we may term *robust counterfactual reliability*: a feature of judgements which *we regard* as sure and solid no matter what circumstances, however speculative or fantastic, are under consideration.

Alright, it may be said. What of it? The bearing of the point for the issues concerning normativity emerges when due notice is taken of a respect in which, in contrast to much of pure mathematics, elementary arithmetic is special. This is the strikingly unconditional applicability of arithmetic to quotidian empirical matters. Much of pure mathematics has, to be sure, fruitful empirical applications. But in a wide range of cases, this application proceeds through an interpretation (perhaps a 'representation theorem') of the mathematical concepts special to the mathematical theory in question in terms of properties and parameters whose primary
p. 460 sphere of application is within the scientific or otherwise empirical field to which ↳ the application is being made. The application of elementary arithmetic is not like that. We learn the concepts of elementary arithmetic precisely by learning to apply them in ordinary practical contexts. It is for that reason that J. S. Mill was able, however misguidedly, to construe arithmetic as comprising a body of well-supported inductive generalisations. There is no gap between the meaning of '7' as it features in number theory and its meaning as learned by a child who is learning to count on her fingers. This is also why Frege took it as part of his brief in the logicist programme for pure arithmetic to account for the content of arithmetical statements in such a way that their applications followed directly from that account. This is exactly what Hume's Principle, so called, achieves. The applications of elementary arithmetic are an *immediate* part of its pure mathematical content.

So, elementary arithmetic, in our customary employment of it, presents the philosopher with a striking combination: of robust counterfactual reliability and immediate applicability. We treat arithmetic as robustly mind-independent but also as needing no lemmas of interpretation in order to have bearing on the empirical world. So treated, it accordingly places constraints on how we conceive the world can behave. So indeed, we would once have said, does Euclidean geometry. But we now know that that claim required a proviso—Euclidean geometry bears directly on the empirical world *provided the space of application is locally Euclidean*. There is no need, we think, for such a proviso when it comes to arithmetic. It is not that if you have seven bananas in a bowl and five apples and no other pieces of fruit, then there will be twelve pieces of fruit altogether *provided that local pieces of fruit are arithmetical*! Everything, actual and possible, is arithmetical.

The result is that no more than an appeal to arithmetic is needed in order to discount certain appearances as misleading no matter how durable they may prove to be. If you carefully and repeatedly count the pieces of fruit, finding seven bananas and five apples and no others, but then on counting all the pieces of fruit together you repeatedly get thirteen, then you can, so we think, completely safely conclude that you miscounted or that the counted groups were not constant in number—even if it obdurately appears otherwise! Even if no one can detect any counting error or fruit joining or exiting the bowl. Perhaps you are going mad. In any event, you can be rightly confident that all is not as it puzzlingly and persistently seems. When results do not 'add up', our response is not at all to be compared to what it has been when paths of light travelling over galactic distances presented non-Euclidean values.

Reflect, therefore, that one absolutely central kind of epistemic normativity belongs to principles that determine what we ought or ought not to think when presented with apparent evidence of certain kinds. The presumed robust counterfactual reliability of elementary arithmetic, combined with its immediate applicability, thus places its principles under this heading. When we apply arithmetic to practical problems, the expectations it generates may indeed not be fulfilled. But when that happens, we are required to look elsewhere p. 461 for the explanation. It is not ↳ an allowable thought that, on this occasion or in these circumstances, arithmetic is a defective guide to reality.

3 Mind-Independence and Realism

So, elementary arithmetical judgements are, in our treatment of them, robustly normative over evidence assessment. Many of them, so we think, are true. And they are counterfactually robust—we think they hold whatever the circumstances and whatever our opinions. Is that enough for normative realism about arithmetic?

I think it should be clear that it is not. Realism should intuitively incorporate some idea, as it is often figuratively expressed, of 'truth not of our making'. But, staying with the metaphor of manufacture, there seems to be no essential solecism in the idea that we might manufacture truths that are to be regarded as holding good even for hypothetical circumstances in which we go in for no such manufacture.

An example of something close to that thought would be our readiness to hold, with Locke, that our actual patterns of colour-experience are implicated in the constitution of the colours alongside the idea that tomatoes would still be red even if we lost, or had never had, the ability visually to distinguish red from green light. Might something of the sort be true even of arithmetic?

It is perhaps not obvious how exactly to debate the matter. We want to give sense to a distinction between forms of mind-independence that are, in a sense, *mind-manufactured* on the one hand and, as it were, true mind-independence on the other—mind-independence of judgements that is imposed purely by the metaphysical nature of the subject matter concerned. My proposal here is that the distinction should turn on the details of the epistemology of the judgements at issue: specifically, that the way to bring the sought-after substance to the idea of truths that *would obtain even if we remained totally innocent of them*, is to make out an epistemology for the judgements in question in which they are disclosed, in the best case, as involving recognition of counterfactual robustness, a justified appreciation precisely that facts are being brought to one's attention which owe nothing to one's attending to them or to the character of one's experience when one does so attend. This (to be sure) vague proposal will structure the discussion to follow.[8]

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4 Basic Arithmetical Knowledge: Proofs by Processes and Pictures

So, what can we say about the actual epistemology of elementary arithmetic? The attitude we take to propositions like 7+5=12 are a creature of teaching and cognitive processing that takes place very early in our lives. But in any case it is clear that no form of linear deductive reasoning is relevant to our present concerns, since it is hardly foreseeable that such reasoning could discharge all assumptions—in which case our question, about the grounds for robust counterfactuality, would effectively simply be referred back to the parent assumptions (whatever they might be) of such reasoning. If we look back at what we actually do in coming to our most basic arithmetical beliefs, is there something to be found to substantiate a realist outlook about their subject matter—something to impose a treatment of them as robustly counterfactually valid?

Let's start by thinking about the kind of thing that very young children can do after they have learned to count groups of objects but before they have mastered the rules of simple arithmetical calculation. Four-year-old Sophie may have no knowledge of addition as an operation with the finite cardinals, but she may nevertheless convince herself that, say, 4+3=7 by working through a diagram; for example

> 1 2 3 4 5(1) 6(2) 7(3)

or by counting out on her fingers. Thus, she might hold up her hands, palms inward, and:

(1) Point successively to her left thumb, saying to herself 'One' as she does so, then point to her left index finger, saying 'Two', then . . . , and then point (now with her left index finger) to her right ring finger and say 'Seven'.

(2) Next, point again to her left thumb and saying 'One', then . . . , and then point to her left ring finger and say 'Four'.

(3) Next, point to her left little finger and say 'One', then . . . , and then point to her right ring finger and say 'Three', and finally conclude:

(4) 'So, four and three makes seven.'

Probably we'd be happy to say, perhaps in the role of proud parents, that she'd done something correctly in following through these simple routines, and that her conclusion was appropriate. But counting is not essential. Even more simply, Sophie may become adept at recognizing the numbers of small groups of things without counting (i.e. by 'subitizing'), so that she can tell at a glance that a group of suitably arranged matchsticks contains exactly five matches. And in that case this figure

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p. 463 may convince her that 2 and 3 make five. And again, we'd probably be happy to allow that her conviction was well conceived.

What, if any, is the significance of these points for the epistemology of elementary arithmetic? Is Sophie indeed getting *knowledge, of* 4+3=7 and 2+3=5 respectively, by such elementary means? If so, knowledge of what propositions exactly—can the last process, for instance, legitimately convince her that two squeaks and three bangs are five sounds? Or is the proposition of which she becomes convinced restricted somehow to objects that are susceptible to a certain kind of spatial array? If so, what *is* that proposition, exactly? Whatever it is, is it a counterfactually robust truth?

These are relatively neglected—and, as it turns out, quite difficult—questions. One reason to take them very seriously—apart from our present concern with normative realism—is that the great foundational programmes in the philosophy of mathematics—the various kinds of structuralism, logicism, and others—although purporting to explain the content and nature of our knowledge of a broad sweep of mathematical subject matters, seem to have very little to say by way of illumination of these most basic phenomena of mathematical conviction. Such views profess to offer an account of the content of mathematical statements and of how they can be known, but their proposals in the latter regard seem on the face of it utterly disconnected from anything that goes on in the processes that actually secure our conviction at the earliest stages of our arithmetical lives. It is tempting to think that no fully satisfying philosophy of mathematics can afford to postpone addressing this disconnection indefinitely.

There is a second, more purely epistemological motive for pursuing the question. Knowledge of robustly counterfactually true propositions cannot, it seems, be empirical. The furthest empirical knowledge can reach into the counterfactual must be constrained by scientific law. But *robust* counterfactuality of the kind we are dealing with in the case of arithmetic precisely transgresses the limits of actual physical law. So we have to be looking for some form of non-empirical—Shock! Horror! —some form of a priori warrant. But even friends of the a priori are currently woefully short of convincing models of the *basic* a priori—of plausible templates for a priori knowledge- or warrant-acquisition achieved other than on the basis of inference from background knowledge, or warrant, already conceived as a priori. The offerings out there on the shelves include various kinds of 'non-inferential faculty' proposals: rational insight, conceptual reflection or, more historically, distillation from the forms of inner and outer sense, together with various still-surviving proposals in the tradition of the broadly conventionalist tendencies of the early to middle 20th century: knowledge of linguistic rules, or of 'tacit conceptions', and the various other inferentialist proposals offered to account—it is never quite explained how—for our presumed knowledge of presumed basic a priori propositions. All of these accounts should address the matter of knowledge of counterfactual robustness. But all are open to familiar
p. 464 problems or limitations, ↳ and all seem prima facie underdeveloped and unconvincing for the range of cases that here interest us, particularly where routines involving some kind of process are involved.[9] One might hope that by paying close attention to the question how conviction is secured in basic arithmetical cases, we can come up with some new models of non-inferential a priori knowledge that may help both to address any climate of scepticism about it and to demystify it a little, as well (perhaps) as drawing attention to what may prove to be significant limitations.

We can dispense with the pre-school stage-setting however. Our question is not really about what children may be legitimately convinced of by the kinds of routines gestured at, but about the cognitive force of the routines themselves. What, if anything, arithmetical is a picture like

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or a process of counting out on the fingers, really empowered to show? What (if anything) additional needs to be brought to bear by the thinker in order for it to show anything *absolutely counterfactually robust*? And what is the cognitive routine one needs to run through in order to appreciate that result? It goes against the grain to deny that such pictures, or processes, are *in any way* evidential. But if they are evidential, what exactly are they legitimately taken to be evidence for and how do they manage to be evidence for it?

5 An Initial Puzzle: The Seemingly Empirical Character of Some of the Process-Proofs

It would be premature to assume that the distinction between arithmetical pictures and arithmetical processes illustrated by the two examples so far given marks any difference in epistemological significance. But let's anyway focus to begin with on simple arithmetical processes, like the apparent verification on the fingers that

> 4+3=7.

There is a clear difficulty with the suggestion that such a routine can produce knowledge a priori. When I run through it, am I not *counting*? But if so, am I not counting (some of) *my fingers*? Do I count my fingers up to the
p. 465
right ring finger ↳ and verify that there are seven, then count again as far as my left ring finger and find that there are four, then count from my left little finger up to my right ring finger and find that there are three? How exactly then do I get to conclude that 7=4+3?

If a priori knowledge of something robustly counterfactual is to be the result, then it seems this cannot be enough. For if I was counting my fingers, then it seems that in concluding that 7=4+3, I would be drawing a conclusion from premises about my fingers each of which I had verified *empirically*, and so—presumably—would know only a posteriori. And then, if I believed the conclusion only because I had inferred it somehow from those premises, my grounds for the conclusion would be at best a posteriori too. How do I get to knowledge of anything stronger than: these four fingers and these three fingers together compose a collection of seven fingers?

Yet from the outside it would certainly look as if I was counting my fingers! And it would certainly look as if I got to the conclusion by inferring it from premises that I had verified by counting my fingers. If that is not what I am doing, what am I doing? And how could it somehow support an a priori warrant for the counterfactually robust arithmetical truth that 7=4+3?

Well, perhaps we might wonder whether counting must always be conceived as an empirical a posteriori method of finding out how many things of some kind there happen to be.

Suppose you ask me how many primes there are up to 20—so I close my eyes and think: '2, that's one; 3, that's two; . . . ; 19, that's eight.' In this case I go through a process that involves keeping count, but the process involves only pure thought, unalloyed by empirical input, and many (at least among those who accept the notion) would be happy to say that I can in this way verify a priori that there are 8 primes less than 20, and verify that that is so as a matter of necessity.

But if we do say that, we need to say what distinguishes the cases in which counting gives only a posteriori knowledge of contingency. For surely when I count the chairs in the third-floor seminar room at NYU and find that there are 17, I don't acquire a piece of a priori knowledge. And similarly—bracketing any noise from the long familiarity of the fact—it is not a priori, still less a counterfactually robust truth, that there are four fingers on my left hand excluding my little finger, . . . and so on.

It is of course contingent how many fingers I have, and not contingent how many primes there are up to 20. We can envisage a routine that combines the shape of the finger-routine with the modalities involved in the routine with the primes. I can execute this routine out loud or in my head: I just say to myself out loud, or just think:

p. 466
'one : one'

'two : two'

'three : three'

'four : four'

'five : one'

'six : two'

'seven : three'

‘So seven is four plus three!’

Here, I imagine that if you asked me *what* I was counting, I’d say: ‘Nothing.’ (It doesn’t seem right to insist that I would be counting the numbers themselves.) I am doing what Paul Benacerraf called intransitive counting.[10] But it is not implausible to suggest that, as far as the ‘mathematically essential’[11] thing is concerned, it is all the same whether I do the fingers routine or the intransitive counting routine: that they are, if you like, different realizations, or tokens, of the same elementary proof type. And if that is right, there is after all no dependence on contingent premises in the fingers routine.

The intransitive counting routine can be, so to speak, purely acoustic, but we can go ‘inside the head’ with it without loss of its epistemic force. And in visual cases too, something similar seems right. I can dispense with the diagram and, eyes closed, just imagine the array

> 1 2 3 4 5(1) 6(2) 7(3)

and accomplish the verification—if that is what it is—that the diagram provides purely via the imagined realization. All this seems a priori in the intended spirit of that notion.

So: the idea that has surfaced is that to understand the evidential force of these simple processes, we need to note that the mathematically essential thing about them is manifest in a number of different kinds of token of essentially the same process, and that its independence of the empirical, contingent features of the concrete cases is shown by its survival in processes ‘in the head’ in which those features are replaced by features that are necessary and constitutive of the relevant objects of thought.

Well, maybe. But this sounds more like the formulation of a perspective to which it would be satisfying to win through than an explanation of anything. How is it possible for episodes in imagination and episodes of concrete external process to have the same epistemic significance? And if it is true that both somehow produce what is in effect the same information, from which an inference then takes us to the p. 467 ↳ arithmetical proposition, what is the nature of that information and what is the nature of that inference?

6 A Second Puzzle: The Ultra-Generality of the Beliefs Induced

There is a second, somewhat daunting constraint on any satisfying account of the content of the knowledge achieved in such cases—if knowledge *is* achieved—that we need to flag up before going any further. However improbable, this is possible: that Sophie works on her fingers as above, but draws only the limited conclusion that four fingers and three fingers make seven fingers. Asked: ‘How about three bananas and four bananas?’ she hesitates, looks unsure, and maybe even says; ‘Hmm, I’ll need to count some bananas to see about that.’ Of course there is *a* question to which that would be a perfectly sensible answer: something like, ‘If you count out four bananas, and then three more, how many will you in fact find if you then count them all together?’ Even then, the question has to be heard as about e.g. the behaviour of *bananas when counted*, or as about the subject’s own *propensities when she counts bananas*, rather than about anything specifically arithmetical. In any case, it seems that the very same routine can also legitimately be received as a verification of something empirical, contingent, and restricted. By contrast, when the process is, as it were, taken arithmetically, the cognitive accomplishment involved is apparently (quite dramatically) *general*. And it is this generality, extending to imaginary and counterfactual cases, that is what the realist needs to underwrite by calling attention to some relevant aspect of the process. It is when we assume that the process is being taken that way that we expect the question about bananas to elicit the answer, ‘Of course.’

You have to suppose that it would take us a long way to our goal if we could give even the rough shape of a plausible model of how the relevant generality might in principle be apprehended. As a first step, though—and to get a sense of how daunting the task is—reflect that there are at least six modes of generalization seemingly involved.

We have already touched on two of them, typified by the transitions from fingers to bananas and from the spatial to the e.g. auditory. For ease of exposition, let's switch to a static picturing case, like the 'matchstick proof' that 2+3=5. Then the fuller catalogue of modes of general significance tacitly ascribed to the configuration looks something like this: we take in the pattern of matchsticks, collectively so configured, and generalize the arithmetical relationship we find there to

(i) the same matchsticks, so configured, at any other time;

(ii) other matchsticks, so configured, at any other time;

(iii) these matchsticks when not so configured;

(iv) other matchsticks when not so configured;

p. 468 (v) ↳ other spatial objects generally, however configured;

(vi) things (par excellence, sounds, say, or days of the week, or abstractions of various kinds) which cannot be spatially configured at all.

A satisfactory explanation of basic knowledge of arithmetic, if it is to address our impression of the absolutely general, counterfactual reach of what is known, has to explain how this (as it may seem) Gadarene rush into generality is somehow brought within epistemically responsible reach.

Of course we are familiar with a least one mode of generalization which is equally far-reaching, at least potentially, yet is transparent and perfectly epistemically responsible: that of *logical* generalization, where we reason from certain of the properties, real or hypothesized, of a specific, or token 'arbitrary' object to the conclusion that it has certain other properties, and then proceed to generalize the entailment. But here, of course, controls are put on the inference to ensure that no collateral condition is tacitly invoked to support the inference that might fail in another case. It is obscure as to which, if any, of the six hurdles above the usual mechanisms of logical generality could possibly be helpful. Logical generality is accomplished by *reasoning* that is designed to avoid exploiting any potentially special features of an item under consideration. But since we have yet to identify any reasoning whatever in the mechanisms of simple arithmetical picture or process proofs in the first place, there is no evident carry-over. *Punkt*.

7 A Proposal: Exploiting Type and Token

So much for stage-setting. Let me now propose what I suggest is the most promising response.

Earlier, I briefly scouted the idea that the counting-on-the-fingers routine for '4+3=7' and the corresponding intransitive counting routine, perhaps executed 'in the head', should somehow be viewed as tokens of essentially the *same process*, and that their mathematically probative character was bound up with that. But I remarked, in effect, that this was closer to a formulation of our problem than to anything amounting to a solution—to a concrete explanation of the epistemological power of picture and process proofs—and set it aside. Maybe that was over-hasty. Let us take a closer look.

We distinguish, familiarly, between type- and token- inscriptions. Here are two tokens of the same type:

> **H** **H**

viz., the letter Aitch in bold upper case Calibri font. Now it's salient that the very notion of a type carries with it
p. 469 the implication of a set of essential characteristics. ↳ Consider 'Any accurate token of the letter "H" in Calibri font[12] contains four right-angles and three lines.' That is a counterfactually robust truth—it holds and is known to hold in any situation in which a token of the letter 'H' in Calibri font is or would be instantiated. But, it is tempting to say, it is an utterly unmysterious example of such. The very formation of any well-conceived notion of a type involves drawing a distinction between features of purported tokens of it that may vary without compromising their status as such and features that may not—features such that nothing will *count* as a token of the type unless it possesses them. And a distinction, thereby made, between the essential, or constitutive, features of tokens of a type—features that they have *qua* tokens of that type, as we like to say—and the rest, is going to subtend a range of truths which we can unproblematically appreciate to be counterfactually robust.

These are hardly controversial observations. Note, though, that they are enough to blow away any wholesale scepticism about (absolute) counterfactual robustness. There are absolutely necessary truths about the type letter 'H'! There are absolutely necessary truths about any type. It is the nature of types to effect a distinction between essential and accidental features of their tokens. And the notion of a type extends, of course, to cover much more than inscriptions, including shapes in general, types of event, types of sequences of types of event, sounds, sequences of sounds, hues, ... So the general point is the following: *any* type is going to be individuated by its instantiation-conditions, and so is going to draw a distinction between those conditions and other conditions which instances of it may or may not satisfy, as it were optionally. That there should be counterfactually robust truths of this character—prototypically, generalizations of the form: 'Any F is G' —is part of the very nature of types, part of their capacity to serve as vehicles of discriminative thought. That there should be such truths is part of the deal in having concepts of types at all.

What about realism? Is it a mind-independent truth that any accurate token of the letter 'H' contains four right-angles and three lines? Is it conventional, of our manufacture, or imposed on us from without? The question impresses as clumsy; one wants to say, 'Neither.' Notice that the question is nothing to do with language or meaning. We are operating at the level of concepts—the level of thought. But the letter 'H' is, it is tempting to say, some kind of creature of thought; it is not something we encounter by directing our thought upon the world. And it is of course conventional that we tie that particular orthographic shape to the eighth letter of the alphabet, give it the name 'Aitch', associate it with the characteristically aspirated sound, and so on. Still, it is no convention that *that* particular orthographic type requires certain characteristics in its tokens. Consider an analogy. Chess, and the chessmen, are some kind of creatures of thought. We invented them. And it is of course
p. 470 conventional that we have a piece (a type-piece) with the powers of ↳ movement and capture of the bishop. But it is not a convention that the *bishop* has those particular powers of movement and capture. If we were to suspend those constraints on the powers of movement and capture of the bishop, there would no longer be bishops in chess. Chess is of our making. That any piece in chess is accorded those particular powers is of our making. That the pieces with those particular powers are called 'bishops' is of our making. That *bishops* have those particular powers is not.

Still, to deny that truths of this character are properly characterized as conventional does not open up any philosophically challenging question about how they may be known—as if in that case knowledge of them would have somehow to be *responsive*, or a matter of tracking. *Bishop*, like *Aitch*, is a type—only now not an orthographic type but a type of chess piece (where that is, loosely, a functional type: it is conventional what they look like, though normal in Western sets that there is some kind of gesture towards a mitred head). And to know what a particular type *is* is to know at least some of what features a token of it has to have, as we have observed. So much is given with a competent concept of the type.

Well, what of any of this? How does it help with our central problem? That problem is to make it intelligible somehow that reflection on simple pictures and processes could lead us to knowledge of counterfactually robust arithmetical truths in particular. And the persistent difficulty has been to explain and justify exactly the *generality* of the known propositions—that they are good not just for the particular picture or process but for any (actual or counterfactual) case; and this, it appears, just on the basis of inspection of and reflection on a single case, real or imagined. So the salient suggestion is that the generality we are trying to understand is, roughly, that implicit in the notion of a *type*: the token, presented to external observation, or to the imagination, is conceived as of a particular type and the generality of the proposition of which we are led to knowledge is the generality that belongs with the very notion of a type. Simple inspection of a token *can* encompass this generality (the suggestion is) when the inspected features are ones that belong, essentially, with the type it tokens.

So that is the core idea of the type–token proposal that, I suggested, is the most promising response to our problem known to me. Is it progress? Yes, and no, I think. 'Yes', in so far as it does seem, in broad outline, that the proposal correctly identifies a principal source of the general significance of these constructions and processes—of their projectibility to other cases and our sense that something has been shown to hold absolutely generally. When we take them to be representative of a type, there is the potential for valid conclusions about essential properties of the type.

But 'no', for two reasons. First, recall that in the basic arithmetical case, any proof that 2+3=5 that can be based on inspection of the matchsticks figure

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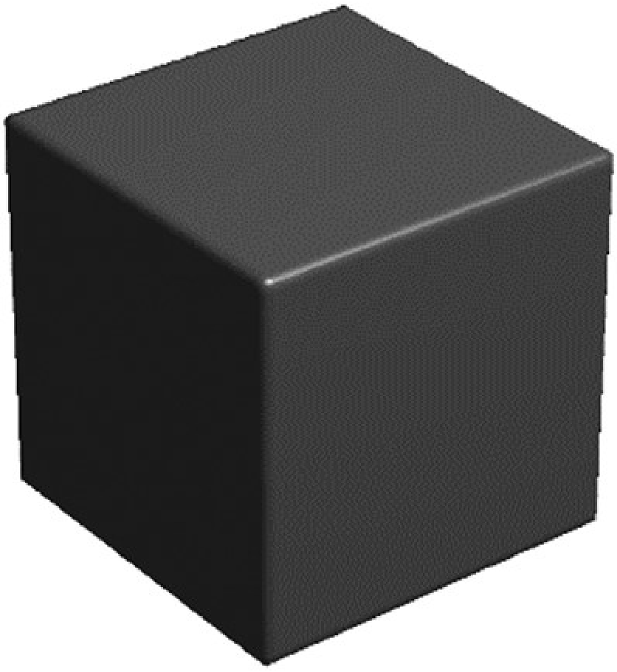
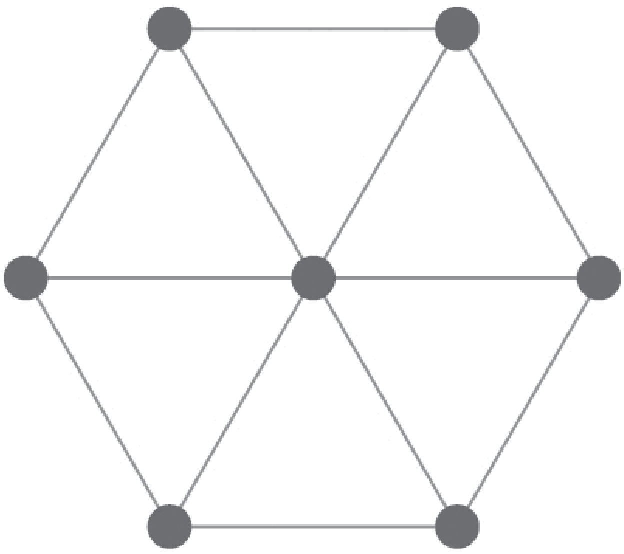
p. 471 has somehow to bring within range not merely any other instance of that configuration, but also objects that are not so configured, and objects that *cannot* be so configured. It has not been made intelligible how that is to be accomplished so long as the notion of a type is closely tied to the paradigms—shapes and patterns—on which we have been concentrating. The point remains that to have any chance of understanding the (remarkable) generality of arithmetic on this model, we will first need to find a legitimate generalization of the notion of a type going way beyond the initial pictorial or process-related illustrations but which yet preserves the possibility of valid registration of its essential characteristics based on scrutiny of those (visual or acoustic) cases.

There is much more to say about that, but the second caveat is anyway the more fundamental. To bring it out, let's put arithmetic to one side and revert to an example where the generality is less spectacular. A geometrical case will serve. Suppose you are asked, How many edges has a cube?—and happen never before to have thought about it. You might happen to know an abstract definition of a cube: say, a three-dimensional solid bounded by six square faces, with three meeting at each vertex, and you might be able to reason deductively directly from that. Well done if so! But more likely you will imagine a cube, maybe like this



and perhaps reflect: ‘Well, I can see nine edges, and the six at the edge of the displayed image are shared with the other side, so there will be three that I cannot see; hence 9+3, so twelve in all.’ Or you might reason that the edges of any two opposite square faces will total eight in number and will be connected one-to-one at the vertices by four more edges; and that will be all the edges—so 8+4, = twelve in all. Or you might think, ‘Well, there are six faces, and each has four edges. But each edge connects two faces—is used twice, as it were. So four times six divided by 2—twelve!’ Or you might project the cube, Necker cube-style, as a 2D hexagon with interior connections like this:



p. 472 and then proceed simply to count the lines connecting the nodes. And no doubt there are other ways of going about it. But what is striking about these and (one would expect) *any* way of going about it that does not amount merely to abstract deduction from an adequate formal definition are the *generalizations* that unavoidably feature in the informal reasoning—for instance: ‘Any cube presented to view in such a way that three of its faces are visible will have three unseen edges’; or ‘Any pair of opposite faces of a cube are connected by four edges’; or ‘Any cube has six faces and each of its edges connects two of them’; or ‘All the faces and edges of a cube can be projected into a two-dimensional figure, thus . . .’ And these generalizations need not have been given as any explicit part of an original characterization of the type *cube*, but nor need they be reasoned to as *propositional* entailments of some anterior characterization. Indeed, there need have been no such anterior characterization—the type may have been conveyed simply by pictures and illustrations. In that case there will be nothing to reason deductively from.

So, to take stock of the Type–Token proposal:

- the notion of a type does indeed incorporate a distinction between essential and inessential features of its tokens;
- that distinction will indeed ground some absolutely counterfactually robust truths; and
- a grasp of (some of) these truths will indeed be part of what is involved in grasping the type.

But: the trouble is that, in the example just reviewed and in the general run of novel cases, our grasp of the type, of its essential features, seems *to surpass*—exactly in the ways that simple picture and process proofs bring out—the information that one might plausibly regard as explicit in a standard explanation of the type; and yet it is not in general *deductively* unpacked from that information. Of course, that way of putting the point is not wholly happy, since in the rough-and-tumble of a normal education, there will in general be no uncontroversial way of fixing on what exactly that information has been. Still, in cases where a type is fixed by a picture or a diagram, as *cube*, or *quintet* maybe, our capacities for the extraction of general propositional information remains (albeit, from an everyday perspective, a completely unremarkable part of ordinary intelligence) philosophically puzzling. The counterfactually robust implications that go with the very notion of a type, and whose epistemology is unproblematic, are only those that are *explicitly* fixed by determining what counts as permissible variation in tokens of the type and what does not. And the problem is that there will be a large range of generalizations, like those exploited in settling, in one of the various ways illustrated, that a cube has 12 edges, which we think a normally intelligent and receptive subject ought to 'get' but which were not made explicit. It takes *insight*, we might say, to grasp the essential characteristics of a type, and the insight is not in general that of foresight of the logical consequences of an explicit characterization.

p. 473 This variation on the matchstick figure:

> | | | | |

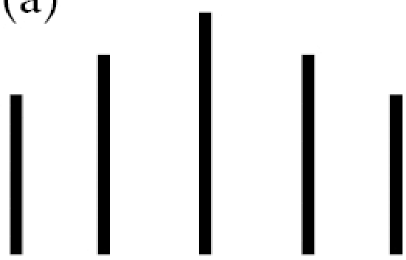
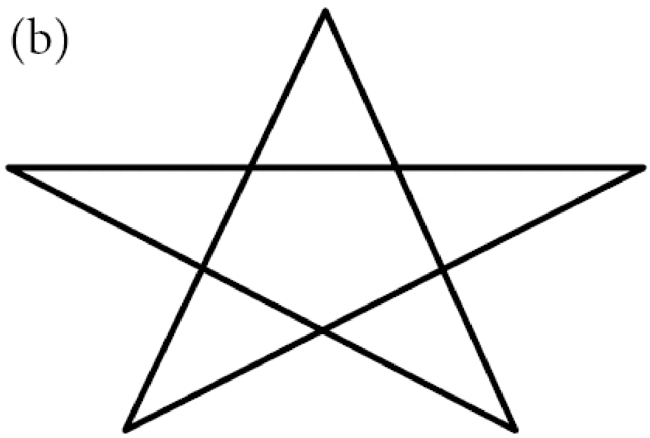
works as a proof, on this model, by presenting a token of the type *quintet*. It will go with that that certain essential characteristics are invoked. Grasping the type will involve having an adequate conception of these characteristics and a sense that they are essential. But how does that get us that 2+3=5? To treat the figure as a verification of (something like) the arithmetical proposition, we need to see that disjoint tokens are also simultaneously presented of *pair* and *trio*, and this not as an incidental matter, like the light grey of the pair, or the dark grey of the trio, but is a matter of the mutual essence, as it were, of all three types. More, we need to see that the relationship allows of generalization to other kinds of configuration of spatially located things, and to the non-spatial too, as I am no doubt boring the reader by repeatedly emphasizing. The picture seems to *unfold*, or *enlarge on*, the essences to bring out a point that, at a very early stage of our intellectual lives, is something we have to learn. We have so far no account of this 'unfolding'.

So the Type–Token proposal does indeed wind up as not much more than a restatement of our problem. The invocation of the notion of a type does seem promising in connection with the sense we have that the pictures and processes in baby picture- and process- proofs are somehow *parametric* and in explaining the generality of their content. But there is still a problem with the spectacular *extent* of the generality operative in arithmetical cases. And the proposal provides no account of our non-deductive acknowledgement of generalizations about types which seem to exceed or enlarge on anything plausibly regarded as simply explicitly written into the type. Such generalizations typically feature in the informal processing of such proofs—indeed, sometimes, as perhaps with the figure above, the acknowledgement of such a generalization seems to be all there is to it.

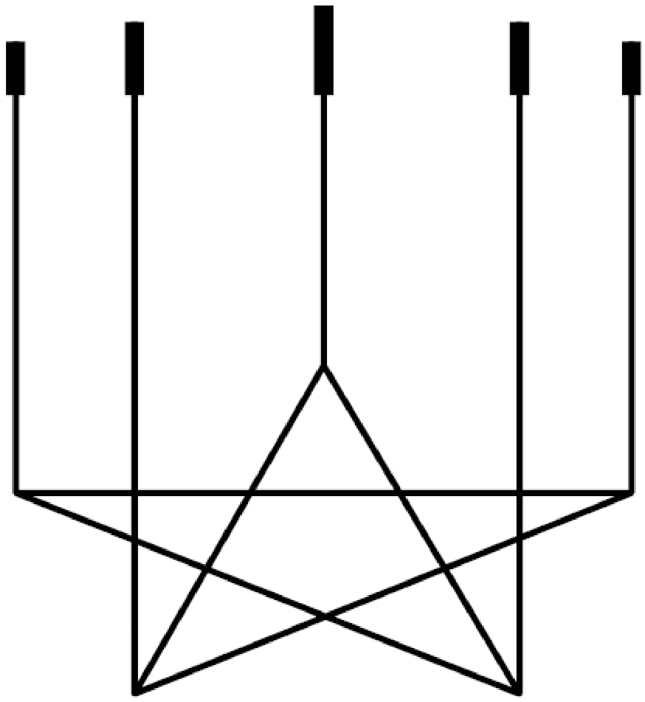
8 Wittgenstein's Deflationary Stance

The foregoing reflections bring us at last alongside one of the central themes of the *Remarks on the Foundations of Mathematics.* A deflationary, non-cognitivist attitude to mathematical necessity pervades the text. Wittgenstein speaks of the effect of proof in general as being to generate, not discovery, but the 'hardening' of a proved proposition into a rule, and of proof as working not by the triggering of any genuinely cognitive *insight* into the obtaining of something ultra (counterfactually) general but by a kind of *persuasion.*[13]

p. 474 A reminder may be useful of the kind of way he tends to write about the issues. The following excerpt is from part I of his text (I'll interpolate interpretative remarks):

> **§25** But how about when I ascertain that this pattern of lines:
>
> (a)
>
> 
>
> is like-numbered with this pattern of angles:
>
> (b)
>
> 
>
> (I have made the patterns memorable on purpose) by correlating them:

(c)



Now what do I ascertain when I look at this figure? What I see is a star with threadlike appendages:

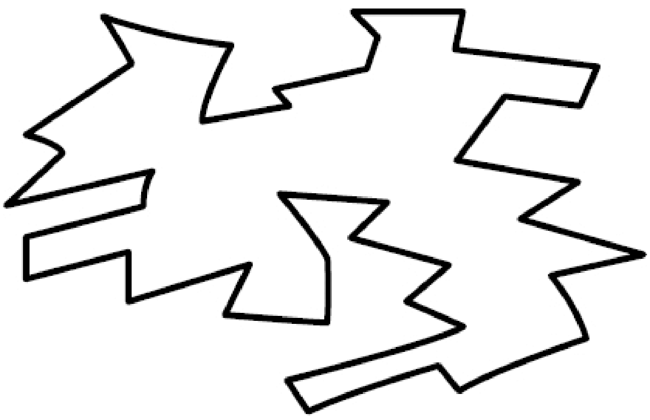
That is, the *empirical* content of what one sees is ‘a star with threadlike appendages.’ How do we elicit any *mathematical* (as Wittgenstein puts it below, *non-temporal*) content out of it? Well, it seems he wants to say, by conferring a certain kind of use on the picture:

> **§26** But I can make use of the figure like this: five people stand arranged in a pentagon; against the wall are wands, like the strokes in (a); I look at the figure (c) and say: ‘I can give each of the people a wand’.

I could regard figure (c) as a schematic *picture* of my giving five men a wand each.

p. 475 He then contrasts regarding the figure as a ‘schematic picture’ in that kind of way with a purely empirical way of taking it:

> **§27** For if I first draw some arbitrary polygon:



> and then some arbitrary series of strokes

> I can find out by correlating them whether I have as many angles in the top figure as strokes in the bottom one. (I do not know how it would turn out.) And so I can also say that by drawing projection-lines I have ascertained that there are as many strokes at the top of figure (c) as the star beneath has points. (Temporally!) In this way of taking it the figure is not like a mathematical proof

—but is more like the case:

> when I divide a bag of apples among a group of people and find that each can have just *one* apple).
>
> I can however conceive figure (c) as a mathematical proof. Let us give names to the shapes of the patterns (a) and (b): let (a) be called a ‘hand’, *H*, and (b) a ‘pentacle’, *P*.

That is, we now introduce *H* and *P* as *types*. And then:

> I have proved that *H* has as many strokes as *P* as angles. And this proposition is once more non-temporal.
>
> **§30** The proposition proved by (c) now serves as a new prescription for ascertaining numerical equality: if one set of objects has been arranged in the form of a hand and another as the angles of a pentacle, we say the two sets are equal in number.
>
> **§31** ‘But isn’t that merely because we have already correlated *H* and *P* and seen that they are the same in number?’—Yes, but if they were so in one case, how do I know that they will be so again now? —‘Why, because it is of the *essence* of *H* and *P* to be the same in number.’

And the *essential* properties of the types, of course, are changeless.

> —But how can you have brought *that* out

—viz. that it is of the *essence* of *H* and *P* to be the same in number—

p. 476

> by correlating them? (I thought the counting or correlation merely yielded the result that these two groups before me were—or were not—the same in number.)

How can the empirical correlation of the two inscribed token figures have brought out that *H* and *P*, introduced as types as above, *essentially* sustain the correlation?

> —‘But now, if he has an *H* of things and a *P* of things, and he actually correlates them, it surely isn’t *possible* for him to get any result but that they are the same in number.—And that it is not possible can surely be seen from the proof.’—But *isn’t* it possible? If, e.g., he—as someone else might say—omits to draw one of the correlating lines. But I admit that in an enormous majority of cases he will always get the same result, and, if he did not get it, would think something had put him out. And if it were not like this the ground would be cut away from under the whole proof. For we decide to use the proof-picture instead of correlating the groups; we do *not* correlate them, but *instead* compare the groups with those of the proof (in which indeed two groups are correlated with one another).

So, Wittgenstein apparently wants to say, when his baby picture-proof convinces someone of an essential connection between the two displayed types, or paradigms, there is no impossibility—of a breakdown of the connection—which it brings us to ‘see’. Rather, we are empirically sure that the connection it illustrates will be confirmed in no end of normal token cases, and that where it is not, we will want to cry ‘foul’ for reasons we can independently attest; and we are thereby moved to *decide* to use it in a certain way. We do not *read off* an essential connection between *H* and *P*. Instead, we are moved to so coordinate the types concerned that it comes to belong to the essence of both to sustain the connection:

> **§32** I might also say as a result of the proof: 'From now on an *H* and a *P* are called "the same in number".'
>
> Or: the proof doesn't *explore* the essence of the two figures, but it does express what I am going to count as belonging to the essence of the figures from now on.—I deposit what belongs to the essence among the paradigms of language.

And similarly, he continues, generalizing, for proof at large:

> The mathematician creates *essences*.

In short, Wittgenstein appears to hold—at least in this passage—that essential relations between types in particular, and mathematical necessity in general, are never properly viewed as matters of recognition and discovery.

p. 477 As is familiar, the posthumous publication of a selection of Wittgenstein's notes on mathematics was not well received.[14] One thing that annoyed the first generation of commentators was the contrast between the perceivedly puerile character of the kind of example of which the above passage is typical and anything one might plausibly regard as representative of interesting mathematics. Wittgenstein seemed to be belittling mathematics—and betraying his ignorance of real mathematics as well. In a review[15] shortly after the publication of the first edition of *RFM*, Paul Bernays complained;

> Instances of proper mathematical proofs, which are not mere calculations, which neither result merely from showing a figure nor proceed formalistically, do not occur at all in this book on the foundations of mathematics, a major part of which treats of the question as to what proofs really are [. . .][16]

while Kreisel characteristically trenchantly dismissed Wittgenstein's text as 'the insignificant product of a sparkling mind'.[17] It should be more evident to us now than it could perhaps be expected to have been to this first generation of commentators that Wittgenstein's central concern was not really with 'the foundations of mathematics' as that phrase was then customarily understood but with the *basic epistemology* of proof. The 'foundations' on which he meant to focus are the actual mechanisms of ground-floor mathematical conviction, rather than the great foundational projects of Frege, Russell, Brouwer, and Hilbert. What he was trying to explore, in examples like the Hand and Pentacle case, is the epistemological status of the simplest forms of conviction of essential, so counterfactually robust connections—exactly our preoccupation here. And what he unmistakably suggests is that no more than his preferred deflationary account is warranted.

9 Cognitivism or Non-Cognitivism?

The question is accordingly whether our reflections on basic arithmetical cases point to anything that can be used to subvert this kind of deflationary attitude to them: that nothing essential is *discovered*, but we are induced to *make* an essential connection that in some sense was not there before. And the answer, I suggest, so far at least, is 'No.' The point at which we have got stuck is exactly that of trying to p. 478 ↳ explain—to describe in a way that is accurate to the intellectual phenomenology and justifies a cognitivist take on the matter—the putative recognitional accomplishment involved. We have, at least so far, no satisfactory model of it, no account that explicates how (for example) drawing on an essentially non-verbal, pictorial grasp of the shape of a cube, someone can recognize that the edges of any two opposite square faces of *any* cube, in whatever circumstances, will be connected one-to-one at the vertices by four more edges; or how by drawing on the pictures of Hand and Pentacle in (a), (b), and (c), we recognize that *any* tokens of them can be put into one-to-one correspondence. Of course these things impress us as super-trivial and beyond obvious. But Wittgenstein is not disputing the phenomenology of obviousness. Were we not prone to such a phenomenology, we wouldn't be inclined to regard these pictures and processes as proofs in the first place. The question is: Why view ourselves as *tracking independent matters* in making such judgements, rather than—as Wittgenstein would seem to prefer —as finding certain kinds of extension of our commitments natural and compelling and shaping our concepts, and our practice with them, accordingly?

We do, I think, most of us, naturally incline to a kind of naïve cognitivism here: that the pictures and processes *demonstrate*—*teach* us about, *draw our attention* to—essential connections which are there anyway, and that they do so in a way that licenses a special certainty and counterfactual generality. But Wittgenstein does enough to show that there is at least prima facie room for a different way of thinking about the matter. The salient question is thus: What in principle should determine which is the better view—our intuitive cognitivism or Wittgenstein's deflationism? Is this a stand-off?

I'll scratch a little at the surface of the problem, and then conclude by reflecting very briefly on the case of ethics.

We have only considered one proposal in any depth. Readers may want to make others. Deflationism's best argument is the want of *any* kind of convincing model even in outline, of how a recognition of absolute counterfactual robustness may be accomplished. Consider therefore the dialectical situation should it prove that none is indeed forthcoming and ask: Why exactly do we need an explanation in the first place? Why should we not be content with the idea that a certain judgmental capacity of ours is purely cognitive even though no account is in prospect of how it accomplishes what we take it to do?

That defensive question is given an edge by the reflection that what is at stake here is presumably a sui generis capacity: a capacity to recognize, a priori, essential counterfactually robust commitments that are neither explicitly axiomatic nor definitional. Thus we cannot assume that an explanation of the workings of the capacity should proceed by way of an assimilation, or subsumption, of the way it operates under something more familiar and better understood—as when the navigational abilities of honey bees are explained by the sensitivity of their neural apparatus to magnetic field lines, or the computational abilities of mathematical prodigies and some autistic subjects are explained by the ascription to them of p. 479 ↳ certain sub-personal recursive routines. The explanation of a sui generis capacity cannot be a special case of the explanation of anything else we do.

However, these considerations tend only towards the thought that, if there indeed *is* a capacity of non-inferential yet productive recognition of counterfactually absolutely robust truth at work in our ratifications of elementary arithmetical picture and process proofs, it might not be reasonable to expect any explanation in more familiar terms of the detail of its working. But how do we discharge the antecedent of that conditional? Absent any account of its working, why believe in the capacity in the first place? Even if it is unreasonable to demand an explanation of its working, ought we not at least to demand some *evidence* that a genuinely cognitive—fact-tracking—capacity is at work?

The realist/cognitivist may respond that ample evidence for that is provided by the near-universal agreement in judgement that these proofs provoke: all (normal) people agree about their probative force and what it is that they show. But this is a flimsy point. The sense of humour is sufficiently widely shared to make comedy a practicable profession, and it is readily conceivable that it might indeed be near enough universally shared—so that professionally successful comedy was easier than it actually is—even though the kinds of process involved were no different from what they actually are. That would not be enough to enforce our regarding it—if it is not already so—as a genuinely cognitive capacity.

Bats have a sui generis capacity to track the positions and movement of objects in their vicinity by echolocation. But we know this only because we have *independent knowledge* of the things that bats are thereby sensitive to—so know that they are getting something right—and have presumably been able to verify what kind of disruption to their abilities is involved if they are prevented from making the relevant noises and receiving echoes. Animal scientists have (or so I imagine) further been able to construct a physiologically attested account of how echolocational sensitivities are realized in their sensory and neural systems, thereby providing a best explanation of how they are able to get the relevant matters right. With basic arithmetical truths, in contrast, we don't so much as get to first base for a project of that kind. For we have no independent check on the matters that, on the cognitivist view, simple picture and process proofs put us in touch with: there is no antecedent body of counterfactually robust fact *that is given to us independently*, of which our responsiveness to basic arithmetical routines could then be verified as enabling us to keep track.

So the scorecard at this point reads as follows. The significance of the case against normative realism about arithmetic provided by the failure to this point of the explanatory project we have been pursuing is qualified by the consideration that there may be a case for saying that, on reflection, it was unreasonable to expect an explanation. But the pro-cognitivist significance of *this* point is qualified in turn by the further consideration that nor do we have any independent evidence that the judgements we are focused upon are getting their targeted subject matter right.

10 Concluding Reflections: What About Ethics?

Let me summarize the main points of our dialectic. When the notion of a normative judgement is appropriately generalized to allow normativity to be grounded in attitude towards rather than the content of a judgement, elementary arithmetical judgement provides (just as Wittgenstein suggests) a paradigm of epistemic normativity, exerting robustly counterfactual control on the proper assessment of perceptual evidence, and hence prima facie incorporating mind-independent truths. But, as I have suggested, converting the 'prima facie' here to 'properly viewed as' requires making out an epistemology of basic arithmetic that is correctly regarded as responsive to 'matters not of our making'; and while that is hardly the clearest mission, our examination of the kinds of elementary pre-calculational processes and pictures by which we are actually convinced leaves the striking counterfactual generality of our basic arithmetical beliefs unexplained. In particular, the model of Type and Token, initially promising, comes up short.

It is of course very discussible whether, should it prove that no other approach fares better, that should dent confidence in a normative realist view of arithmetic. According to important and influential earlier work by one of the editors of this volume,[18] however, it ought to do so, since what is being presented is a certain conception of the metaphysical character of arithmetical truth in the absence of any satisfactory account of how we might justifiably credit ourselves with the ability to know truths concerning that subject matter.

The dialectical challenge is sharpened by the fact that there is, as I briefly hinted earlier, no easy transition from counterfactual robustness to realism provided there are other, non-cognitivist strategies for explaining our treatment of the truths of the relevant region of thought as counterfactually robust that do not involve our cognitive appreciation of their independently being so. And for logic and mathematics at least there is at least one salient such strategy. Wittgenstein wrote

> I say, however: if you talk about essence—you are merely noting a convention. But here one would like to retort: there is no greater difference than that between a proposition about the depth of the essence and one about—a mere convention. But what if I reply: to the depth that we see in the essence there corresponds the deep need for the convention. (*RFM* I, 74)

If we are to go in for reasoning about what may be or are unactualized scenarios—and such reasoning is of course absolutely constitutive of practical rationality—and to share and coordinate our findings, we will need p. 481 to be able to rely on principles of reasoning that are taken to hold good beyond the actual. 'The deep need for ↳ the convention' may be better put as: the deep need for a shared body of inferential procedures that we confidently mutually accept as apt for this role. The broad upshot of our discussion is that there is no evident reason why such an assurance must always be based on a cognitive ground, and a strong prima facie case that it need not be.

Coda: A thought about the apparent counterfactual robustness of (some) values.

How much if any of the foregoing discussion transposes to the case of morality? It has always seemed to me that the epistemology of fundamental moral conviction, compared to that of elementary arithmetic, is darker yet. There is nothing that underwrites fundamental moral convictions in the way that the kinds of baby processes and pictures we have been concerned with generate our confidence in elementary arithmetic. Yet, as the literature makes clear, many are in sympathy with the idea that basic moral convictions are indeed counterfactually robust at least over variation in our convictions—that the standards they encode would be valid even in circumstances where they ceased to, or never did, exert any pull on us—whether or not, like our arithmetical convictions, they are also robust over variation in other matters.

(Parenthetically, in fact, there is a tension in that separation. For if, for example, there are possible counterfactual circumstances in which, say, its being good to help the needy would no longer have its actual force, would not a hypothetical subject who appreciated the morally etiolating force of those circumstances be a counterexample to the notion that the principle would be in good standing no matter what anyone thought? Can mind-independence, as standardly formulated, avoid escalation into a more thoroughgoing counterfactual robustness? That seems to me an interesting question that I won't here attempt to answer.)

My own view is that the absolute (or the extent of the) counterfactual robustness of even the most basic moral standards is unclear. It is among the issues raised, for example, by the persisting moral sensibilities of The Boy in Cormac McCarthy's *The Road*— his wanting, for example, to share their dwindling food stock—from one perspective, utterly pointlessly—with the starving Old Man that he and his father encounter. But even if we sympathize, it is, just as in the case of arithmetic, a further step to conclude that the counterfactual robustness of at least some moral values is sourced in a normative reality to which we have some form of responsive cognitive access. Rather, as with arithmetic and logic, there are various directions for a possible alternative non-cognitivist account.[19] One thought I myself find telling in this context comes in response to an instance of the Euthyphro contrast: the thought that (some) moral value contrasts intuitively with (say) comedic, gustatory, and even refined aesthetic value more generally, in the sense that those latter values are grounded in our p. 482 responses to their instances whereas moral value ↳ is not. It is in our characteristic affective reactions to good food, good humour, and fine art that an explanation is to be found of *why* there are values of these kinds at all, whereas the corresponding thought about morals impresses intuitively as a misunderstanding of the kind of value involved: examples of courageous, selfless, and fair conduct may indeed inspire warm feelings of approval and other positive emotions, but it is not on their propensity to do so that their value is based.

However, this (if sound) is merely a point of contrast between the *concept* of moral value and the *concepts* of values of other kinds. While it may provide a platform for ordinary thought to transition to the idea that the extension of the former is independent of our affective propensities and hence counterfactually robust through change in the patterns they exhibit, philosophical thought should be more circumspect. For nothing immediately follows about moral epistemology, nothing to substantiate the idea that, as moral realism requires, our fundamental moral judgements should be seen as responsive to values 'not of our making'.

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Notes

Footnotes

1 Henceforward *RFM*. I shall be referring to the 3rd edition (1981).

2 'For what does a proposition's '*being true*' mean? 'P' *is true* = P. (That is the answer.)' *RFM*, Appendix III, §6

3 Wright (1992).

4 To anticipate: one corollary of the discussion will be that an acceptance of the mind-independence of a region of normative thought, when understood as a kind of counterfactual robustness, stops well short of a commitment to realism, in the intuitively intended spirit, about its subject matter.

5 Mark Schroeder for example (2007) takes it that normative properties and relations constitutively involve—have to be analysed in terms of—*reasons*. Paul Boghossian (2005 and 2008) proposes that a judgement is normative just in case 'it is constitutive of our understanding' of it that it 'implies oughts'. Kurt Sylvan (2018) proposes three paradigms of normative concept: the deontic (what is permissible, right, or wrong), the evaluative ('thin' properties like goodness, and thick ones like 'gloriousness and 'grossness') and a third group (which he terms the 'hypological'), comprising concepts like praiseworthiness, blameworthiness, and excusability, and then characterizes normative concepts in general as comprising those included among or analysable in terms of the members of this list. Paul Horwich (2018) seems to concur with Boghossian about the centrality of 'ought', although tentative about the character of the 'ought' that may be involved and wary of claims about what is constitutive of understanding.

6 Most notably in the Appendix III on Gödel's Theorem. But see also III, §25–6 and 28, and IV, §§13ff. Compare the *Lectures on the Foundations of Mathematics* (Diamond 1976) at pp. 33, 55, 70, 98–9, 134–5, 246, and 268.

7 From Louise Hanson (2018) at p. 39. Similar expressions may be found in Walter Sinnott-Armstrong (1987) at p. 263, David Brink (2001) at p. 154, Russ Shafer-Landau (2003) at p. 2, and in many other places.

8 Sections 4–8 of the discussion to follow draw extensively on ideas that were presented in my Petrus Hispanus lectures in Lisbon in 2018, for a full version of which see Wright (2023).

9 Not that efforts have not been made. It would take us too far afield to review details, but for at least one redoubtable attempt, see chapter 4, 'Necessity', of Christopher Peacocke's (1999). My own reasons for regarding Peacocke's intricate attempt as coming well short are detailed in Wright (2001)

10 In Benacerraf (1965).

11 I intend the quotes to remind the reader of Wittgenstein's use of this phrase, also in quotation marks, in *RFM* I at §36.

12 I'll omit the reference to upper-case Calibri font from here on.

13 See for instance *RFM* IV, 30.

14 It didn't help that in the first edition the Editors omitted to include much material, especially the text that became section VI, on rule-following, in later editions, that sheds crucial light on the interpretation of the whole.

15 Bernays (1959).

16 Bernays (1959) at p. 2.

17 Kreisel (1958), concluding remark.

18 Peacocke (1999: ch. 1, 'The Integration Challenge').

19 This is naturally a key thesis for any general 'quasi-realist' approach to ethics.