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What Was Frege's Mistake?

"How did the serpent of inconsistency enter Frege's paradise?" asks Michael Dummett at the start of chapter 17 of *Frege: Philosophy of Mathematics*. And in the final chapter he suggests an answer: that Frege's major mistake – the key to the collapse of the project of *Grundgesetze* – consisted in

> … his supposing there to be a totality containing the extension of every concept defined over it; more generally [the mistake] lay in his not having the glimmering of a suspicion of the existence of indefinitely extensible concepts.[1]

The diagnosis is repeated in the essay, "What is Mathematics About?", where Dummett writes that

> Frege's mistake … lay in failing to perceive the notion [of a value-range] to be an indefinitely extensible one, or, more generally, in failing to allow for indefinitely extensible concepts at all.[2]

Now, claims of the form,

> Frege fell into paradox because…….

are notoriously difficult to assess even when what replaces the dots is relatively straightforward. Paradoxes of any depth are usually complex and seldom involve moves that, once exposed, allow of straightforward identification as clear-cut "mistakes". The paradox attending Basic Law V is no exception. Diagnostic offerings have included –

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1 Dummett 1991, p. 317.
2 Dummett 1993, p. 441.

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(A) *Unrestricted quantification*: Frege fell into paradox because he allowed himself to quantify over a single, all-inclusive domain of objects (Russell, Dummett).

(B) *Impredicative objectual quantification*: Frege fell into paradox because he allowed himself to define courses-of-values using (first-order) quantifiers ranging over those very courses-of-values (Russell, Dummett).

(C) *Impredicative higher-order quantification*: Frege fell into paradox because he allowed himself to formulate conditions on courses-of-values using (higher-order) quantifiers ranging over those very conditions (Russell, Dummett).

(D) *Inflation*: Frege fell into paradox because he adopted an axiom – Basic Law V – which is *inflationary*, i.e. defines its proper objects by reference to an equivalence on concepts that partitions the higher-order domain into too many cells (Boolos, Fine).

And while it is indeed clear that Frege did do all these things – and prior to that, clear, or clear enough, what it *is* to do them – the diagnoses presented are nevertheless problematic. Contra (A), for example, there are multiple instances where unrestricted (objectual) quantification seems both intelligible and essential to the expression of the full range of our thoughts. Contra (B) and (C), while impredicative quantification of both first and higher orders is indeed essential to the generation of the paradox, it is also essential to a range of foundational moves in classical mathematics and, in so far as it may seem objectionable, the objections seem more properly epistemological than logical. Contra (D), there is no straightforward connection, in a higher-order setting, between unsatisfiability and inconsistency; and it is salient in any case that the actual derivation of the contradiction from Frege's axiom nowhere implicitly depends upon an assumption of the classical range of the second-order variables but would go through on, for example, a substitutional interpretation of second-order quantification. However with Dummett's quoted proposal:

(E) Frege fell into paradox because he didn't have even a glimmering of a suspicion of the existence of indefinitely extensible concepts, matters may seem yet worse. This diagnosis may seem not to get so far as proposing *any* definite account of Frege's "colossal blunder" (as Dummett elsewhere characterises it[3]) at all, even a controversial one. What exactly did Frege do, or fail to do, because he failed to reckon with the indefinite extensibility of *extension* or *course of values*? What indeed exactly *is* indefinite extensibility? The notion continues to be met with the kind of scepticism which George Boolos espoused when he roundly rejected Dummett's diagnosis, opining that it was "*To his credit*, [that] Frege did not have the glimmering of a suspicion of the existence of indefinitely extensible concepts" [my emphasis].[4]

Indefinite extensibility has been connected in recent philosophy of mathematics with many large issues, including not just the proper diagnosis of the paradoxes, but the legitimacy of unrestricted quantification, the content of quantification (if legitimate at all) over certain kinds of populations, the legitimacy of classical logic for such quantifiers, the proper conception of the infinite, and the possibilities for (neo-)logicist foundations for set-theory. But my project here must be limited: I shall first address a problem that obscures the usual intuitive characterisations of the notion of indefinite extensibility, and offer thereby what I believe to be the correct characterisation of the notion.[5] En passant, we shall review some issues about the "size" of indefinitely extensible concepts. Finally we will scrutinise the connections of the notion as characterised with paradox. A full enough plate.

1 The intuitive characterisations and the problem of circularity

Dummett's conception of indefinite extensibility, and the suggestion that it is playing some kind of devil's part in the paradoxes, is of course anticipated in Russell. Following an examination of the standard paradoxes, the latter's [1906] concludes:

> … the contradictions result from the fact that … there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any

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3 Dummett 1994, p. 243.

4 Boolos 1998, at p. 224. I should observe, though, that, in context, Boolos is assuming that an indefinitely extensible concept comes with a prohibition on unrestricted quantification over its instances – something that Dummett repudiates in his response.

5 Here I draw on Shapiro and Wright 2006.

> class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* of the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.

For comparison, Dummett [1993, p. 441] writes that an

> *indefinitely extensible* concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it.

According to Dummett, an indefinitely extensible concept *P* has a "principle of extension" that takes any definite totality *t* of objects each of which has *P*, and produces an object that also has *P*, but is not in *t* (see also Dummett [1991, pp. 316–319] where he cites the above passage from Russell).

But what does "definite" mean in that? Presumably a concept *P* is *Definite* for Dummett's purpose in those passages just if it is not indefinitely extensible! If so, then Dummett's remarks won't do as a definition, even a loose one, since they appeal to its complementary "definite" to characterise what it is for a concept to be indefinitely extensible. And Russell, of course, does no better by speaking unqualifiedly of "any class of terms all having such a property", since he is taking it for granted that classes, properly so regarded, are "wholes" or "have a total" – that is, presumably, are Definite.

Notice that it would not do just to drop any reference to definiteness, or an equivalent, in the intuitive characterisation. If the suggestion had been, for example, that an

> *indefinitely extensible* concept is one such that, for any given totality all of whose members fall under the concept, we can, by reference to that totality, characterise a larger totality all of whose members fall under it,

then the usual suspects would fail the test – if we took *set*, for instance, as the target concept and then picked as the first mentioned "totality" simply the sets themselves, there would be no "larger" totality of sets to extend into. And if we then stipulated that attention should be restricted to *proper* sub-totalities, then all concepts would pass the test.

This problem of implicit circularity in the intuitive characterisation of indefinite extensibility is a serious one. Indeed, it is the major difficulty in forming a clear idea of the notion, and one I propose to solve here. But it would be premature to lose confidence in the notion of indefinite extensibility because of it. The three concepts targeted by the classic set-theoretic paradoxes – Burali, Cantor, and Russell – surely present a salient common pattern:

(1) *Ordinal*. Think of the ordinals in an intuitive way, simply as order-types of well-orderings. Let O be any Definite collection of ordinals. Let O' be the collection of all ordinals smaller than or equal to some member of O. O' is well-ordered under the natural ordering of ordinals, so has an order-type – γ. So γ is itself an ordinal. Let γ' be the order-type of the well-ordering obtained from O' by tacking an element on at the end. Then γ' is an ordinal number, and γ' is not a member of O. So *ordinal number* is indefinitely extensible.[6]

(2) *Cardinal*. Let C be any Definite collection of cardinal numbers. Assign to each of its members a set of that exact cardinality, and form the union of these sets, C'. By Cantor's theorem, the collection of subsets of C' is larger than C', so larger than any cardinal in C. So *cardinal number* is indefinitely extensible.

(3) *Set/class*. Dummett writes

> Russell's concept *class not a member of itself* provides a beautiful example of an indefinitely extensible concept. Suppose that we have conceived of a class C all of whose members fall under the concept. Then it would certainly involve a contradiction to suppose C to be a member of itself. Hence, by considering the totality of the members of C together with C itself, we have specified a more inclusive totality than C all of whose members fall under the concept *class not a member of itself*.[7]

Observe that it follows that *set* itself is indefinitely extensible, since any Definite collection – set – of sets must omit the set of all of its members that do not contain themselves.

To be sure, the argumentation involved in these cases is not completely incontestable. Someone could challenge the various set-theoretic principles (Union, Replacement, Power-set, etc.) that are implicitly invoked in the constructions, for instance. But I think it reasonable to agree with Russell and Dummett that the examples do exhibit some kind of "self-reproductive" feature which the notion of indefinitely extensibility gestures at. The question is whether we can give a more exact, philosophically robust characterisation of it.

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6 As Dummett [1991, p. 316] puts it, "if we have a clear grasp of any totality of ordinals, we thereby have a conception of what is intuitively an ordinal number greater than any member of that totality. Any [D]efinite totality of ordinals must therefore be so circumscribed as to forswear comprehensiveness, renouncing any claim to cover all that we might intuitively recognise as being an ordinal."

7 Dummett 1993, at p. 441.

2 Indefinite extensibility and the ordinals: Russell's Conjecture and 'small' cases

We can make a start by following up on a suggestion of Russell himself. Russell [1906, p. 144] wrote that it "is probable" that if *P* is any concept which demonstrably "does not have an extension", then "we can actually construct a series, ordinally similar to the series of all ordinals, composed entirely of terms having the concept *P*". The conjecture is in effect that if *P* is indefinitely extensible, then there is a one-to-one function from the ordinals into *P*.

If Russell is right, then any indefinitely extensible concept determines a collection at least as populous as the ordinals – so, one might think, surpassing populous! And in that case one might worry whether the connection made by Russell's Conjecture is acceptable. For Dummett at least has characteristically taken it that both the natural numbers and real numbers are indefinitely extensible totalities in just the same sense that the ordinals and cardinals are, with similar consequences, in his opinion, for the understanding of quantification over them and the standing of classical logic in the investigation of these domains. Moreover in the article [Dummett 1963] which contains his earliest published discussion of the notion, Dummett argues that the proper interpretation of Gödel's incompleteness theorems for arithmetic is precisely to teach that *arithmetical truth* and *arithmetical proof* are also both indefinitely extensible concepts – yet neither presumably has an even more than countably infinite extension, still less an ordinals-sized one. (The ordinary, finitely based language of second-order arithmetic presumably suffices for the expression of any arithmetical truth.) It would be disconcerting to lose contact with perhaps the leading modern proponent of the importance of the notion of indefinite extensibility so early in the discussion. But then who is mistaken, Russell or Dummett?[8]

The issue will turn out to be important for the proper understanding of indefinite extensibility. To fix ideas, consider the so-called Berry paradox, the

8 It is relevant to recall that Russell [1908] himself, in motivating a uniform diagnosis of the paradoxes, included in his list of chosen examples some at least where the "self-reproductive" process seems bounded by a relatively small cardinal. For instance the Richard paradox concerning the class of decimals that can be defined by means of a finite number of words makes play with a totality which, if indeed indefinitely extensible, is at least no greater than the class of decimals itself, i.e. than $2^{\aleph_0}$. Was Russell simply unaware of this type of example in 1906, when he proposed the conjecture discussed above? Or did he not in 1906 regard the Richard paradox and others involving "small" totalities as genuine examples of the same genre, then revising that opinion two years later?

paradox of "the smallest natural number not denoted by any expression of English of fewer than 17 words". Here is a statement of it. Define an expression *t* to be *numerically determinate* if *t* denotes a natural number and let *C* be the set – assuming there is one – of all numerically determinate expressions of English. Consider the expression: "The smallest natural number not denoted by any expression in *C* of fewer than 17 words". Assume that this is a numerically determinate expression of English. Then contradiction follows from that assumption, the assumption that the set *C* exists, and the empirical datum that *b* has 16 words (counting the contained occurrence of '*C*' as one word).

The analogy with the classic paradoxes may look good, a principle of extension seemingly inbuilt into a concept leading to aporia when applied to a totality supposedly embracing all instances of the concept. But, as emerges if we think the process of "indefinite extension" through, there are complications.

To see why, let an initial collection, *D*, consist of just the ten English numerals, "zero" to "nine". Count '*D*', so defined, as part of English, and consider "the smallest natural number not denoted by any member of *D* of fewer than 17 words". Call this 16-worded expression "W_1". Its denotation, clearly, is 10. W_1 is a numerically determinate expression of English, but not in *D*. Let *D*1 be $D \cup \{W_1\}$. Count '*D*1' as an English one-word name. Now repeat the construction on *D*1, producing W_2. Let *D*2 be $D1 \cup \{W_2\}$. Count '*D*2' as an English one-word name. Do the construction again. Keep going ...

How far can you keep going? Well, not into the transfinite. For reflect that 0 to 9 are all denoted by single-word members of *D*; 10 is denoted by the 16-worded "the smallest natural number not denoted by any member of *D* of fewer than 17 words"; 11 is denoted by the 16-worded "the smallest natural number not denoted by any member of *D*1 of fewer than 17 words"; 12 is denoted by the 16-worded "the smallest natural number not denoted by any member of *D*2 of fewer than 17 words"; and so on. So every natural number is denoted by some expression of English of fewer than 17 words. So the "the smallest natural number not denoted by any expression in *C* of fewer than 17 words" *has no reference* – and hence is not a numerically determinate expression after all, contrary to the assumptions of the paradox.

This result does not dissolve the Berry paradox, since it depends on assumptions about English – specifically, that it may be reckoned to contain all the series of names, *D, D1, D2*, etc., and that these can be reckoned to be one-word names – which may be rejected. The point I am making, rather, is that, *when* the relevant assumptions about what counts as English are allowed, the construction shows that while there is indeed a *kind* of indefinite extensibility about the concept, *numerically determinate expression of English*, it is a *bounded* indefinite extensibility, as it were: indefinite extensibility up to a limit – in this case the first transfinite ordinal, ω. When the limit is reached, the result of the construction is a (presumably) *definite* collection of entities that does not in turn admit of extension by the original operation. So the targeted concept will not be *indefinitely* extensible, at least not in the spirit of Dummett's and Russell's intuitive characterisations.

Consider another example. As noted above, Dummett [1963] contends that Gödel's incompleteness theorem shows that *arithmetical truth* is indefinitely extensible. That should mean that given any Definite collection C of arithmetical truths, one can construct a truth – the Gödel sentence for C – that is not a member of C. This is apt to impress as a problematic claim, at least if sethood suffices for Definiteness, and if the former concept is understood as governed by its classical mathematics. For following Tarski, we may give a straightforward explicit definition of *arithmetical truth*. Assuming a well-defined set of arithmetical sentences, it should then follow by the *Aussonderungsaxiom* that there is a set – a Definite collection – of all arithmetical truths. But there is no "Gödel sentence" for this set.

Still, it is clear enough what Dummett has in mind. It is straightforward to initiate something that looks like a process of "indefinite extension". Just let A_0 be the theorems of some standard axiomatisation of arithmetic. For each natural number n, let A_{n+1} be the collection A_n together with a Gödel sentence for A_n. Presumably, if A_n is Definite, then so is A_{n+1}, and, of course, A_n and A_{n+1} are distinct. Unlike the case of the Berry paradox, this construction can indeed be continued into the transfinite. Let A_ω be the union of A_0, A_1, ... Arguably, A_ω is Definite. Indeed, if A_0 is recursively enumerable, then so is A_ω. Thus, we can obtain $A_{\omega+1}$, $A_{\omega+2}$, ... and so on. Then we take the union of those to get $A_{2\omega}$, and onward, "Gödelising" all the way.

On the usual, classical construal of the extent of the ordinals, however, this process too cannot continue without limit, but must "run out" well before the first uncountable ordinal. Let λ be an ordinal and let us assume that we have obtained A_λ. The foregoing construction will take us on to the next set $A_{\lambda+1}$ only if the collection A_λ has a Gödel sentence. And that will be so only if A_λ is *recursively axiomatisable*. But clearly it cannot be the case that for every (countable) ordinal λ, A_λ is recursively axiomatisable. For there are uncountably many countable ordinals but only countably many recursive functions.

3 Indefinite extensibility explicated

Let's take stock. Russell's Conjecture, that indefinitely extensible concepts are marked by the possession of extensions into which the classical ordinals are injectible, still stands. At any rate some apparent exceptions to it, like *numerically determinate expression of English* (when "English" is understood to have the expressive resources deployed above) and *arithmetical truth*, are not really exceptions. For the principles of extension they involve are not truly *indefinitely* extensible but stabilise after some series of iterations isomorphic to a proper initial segment of the ordinals – at least if the ordinals are allowed their full classical extent.

That said, though, the point remains that Russell's Conjecture, even should it be extensionally correct, is certainly not the kind of characterisation of indefinite extensibility we should like to have. If Russell's Conjecture were the best we could do, it would be a *triviality* that the ordinals themselves are indefinitely extensible. What is wanted is a perspective from which we can explain *why* Russell's Conjecture is good, if indeed it is – equivalently, a perspective from which we can characterise exactly what it is about *ordinal* that *makes* it the paradigm of an indefinitely extensible concept.

So let's step back. An indefinitely extensible totality *P* is intuitively unstable, "restless", or "in growth". Whenever you think you have it safely corralled in some well-fenced enclosure, suddenly – hey presto! – another fully *P*-qualified instance pops up outside the fence. The primary problem in clarifying this figure is to dispense with the metaphors of "well-fenced enclosure" and "growth". Obviously a claim is intended about sub-totalities of *P* and functions on them to (new) members of *P*. But, as we observed, the intended claim does not concern *all* sub-totalities of *P*: we need to say for *which kind* of sub-totalities of *P* the claim of extensibility within *P* is being made. If we could take it for granted that the notion of indefinite extensibility is independently clear and in good standing and picks out a distinctive type of totality, then we could characterise the relevant kind of sub-totality exactly as Dummett did – they are the sub-totalities that are, by contrast, *Definite*. For the indefinite extensibility of a totality, if it consists in anything, precisely consists in the fact that any Definite sub-totality of it is merely a *proper* sub-totality. But at this point the clarity and good standing of the notion of infinite extensibility may not be taken for granted.

Here is a way forward. Let us, at least temporarily, finesse the "which sub-totalities?" issue by starting with an explicitly relativised notion. Let *P* be a concept of items of a certain type τ. Typically, τ will be the (or a) type of individual objects. Let Π be a higher-order concept – a concept of concepts of type τ items. Let us say that *P* is *indefinitely extensible with respect to* Π if and only if there is a function *F* from items of the same type as *P* to items of type τ such that if *Q* is any sub-concept of *P* such that Π*Q* then

(1) *FQ* falls under the concept *P*,
(2) it is not that case that *FQ* falls under the concept *Q*, and
(3) Π*Q′*, where *Q′* is the concept instantiated just by *FQ* and by every item which instantiates *Q* (i.e., $\forall x[Q'x \equiv (Qx \vee x = FQ)]$; in set-theoretic terms, *Q′* is $(Q \cup \{FQ\})$).

Intuitively, the idea is that the sub-concepts of *P* of which Π holds have no maximal member. For any sub-concept *Q* of *P* such that Π*Q*, there is a proper extension *Q′* of *Q* such that Π*Q′*.

This relativised notion of indefinite extensibility is quite robust, covering a lot of different examples. Here are three:

(*Natural number*) *Px* iff *x* is a natural number; Π*Q* iff the *Q*s (i.e the instances of *Q*) are finite in number; *FQ* is the successor of the largest *instance of Q*. So *natural number* is indefinitely extensible with respect to *finite*.

(*Real number*) *Px* iff *x* is a real number; Π*Q* iff the *Q*s are countably infinite. Define *FQ* using a Cantorian diagonal construction. So *real number* is indefinitely extensible with respect to *countable*.

(*Arithmetical truth*) *Px* iff *x* is a truth of arithmetic; Π*Q* iff the *Q*s are recursively enumerable. *FQ* is a Gödel sentence generated by the *Q*s. *FQ* is a truth of arithmetic and is not one of the *Q*s. So *arithmetical truth* is indefinitely extensible with respect to *recursively enumerable*.

And naturally the three principal suspects are covered as well:

(*Ordinal number*) *Px* iff *x* is an ordinal; Π*Q* iff the *Q*s exemplify a well-ordering type, γ (which since *Q* is a sub-concept of *ordinal*, they will) *FQ* is the successor of γ. So *ordinal number* is indefinitely extensible with respect to the property of *exemplifying a well-ordering type*.

(*Cardinal number*) *Px* iff *x* is a cardinal number; Π*Q* iff the *Q*s compose a set. *FQ* is the power set of the union of a totality containing exactly one exemplar set of each *Q* cardinal. So *cardinal number* is indefinitely extensible with respect to the property of *composing a set*.

(*Set*) *Px* iff *x* is a set; Π*Q* iff the *Q*s compose a set. *FQ* is the set of *Q*s that are not self-members. So *set* is indefinitely extensible with respect to the property of *composing a set*.

The relativised notion of indefinite extensibility should impress as clear enough, but it does not, of course, shed any immediate philosophical light on the paradoxes. Our goal remains to define an unrelativised notion of indefinite extensibility that still covers *ordinal number*, *cardinal number*, and *set* but somehow illuminates why they are associated with paradox while *natural number*, *real number* and *arithmetical truth* are not. So what next?

Three further steps are needed. Notice to begin with that the listed examples sub-divide into two kinds. There are those where – helping ourselves to the classical ordinals – we can say that some ordinal λ places a lowest limit on the length of the series of Π-preserving applications of *F* to any *Q* such that Π*Q*. Intuitively, while each series of extensions whose length is less than λ results in a collection of *P*s which is still Π, once the series of iterations extends as far as λ the resulting collection of *P*s is no longer Π, and so the "process" stabilises. This was the situation noted with *numerically determinate expression of English* in our discussion of the Berry paradox, and is also the situation of the first three examples above. But it is not the situation with the principal suspects: in those cases there is no ordinal limit to the Π-preserving iterations. With *ordinal number*, this is obvious, since the higher-order property Π in that case just is the property of having a well-ordering type. Indeed, let λ be any ordinal. Then the first λ ordinals have the order-type λ and so they have the property. The "process" thus does not terminate or stabilise at λ. With *set* and *cardinal number*, we get the same result if we assume that for each ordinal λ, any totality that has order-type λ is a set and has a cardinality.

Let's accordingly refine the relativised notion to mark this distinction. So first, for any ordinal λ say that *P* is *up-to-λ-extensible with respect to* Π just in case *P* and Π meet the conditions for the relativised notion as originally defined but λ places a limit on the length of the series of Π-preserving applications of *F* to any sub-concept *Q* of *P* such that Π*Q*. Otherwise put, λ iterations of the extension process on any Π*Q* "generates" a collection of *P*s which form the extension of a non-Π sub-concept of *P*. Next, say that *P* is *properly indefinitely extensible* with respect to Π just if *P* meets the conditions for the relativised notion as originally defined and there is no λ such that *P* is up-to-λ-extensible with respect to Π. Finally, say that *P* is *indefinitely extensible* (simpliciter) just in case there is a Π such that *P* is properly indefinitely extensible with respect to Π.

My suggestion, then, is that the circularity involved in the apparent need to characterise indefinite extensibility by reference to *Definite* sub-concepts/collections of a target concept *P* can be finessed by appealing instead at the same point to the existence of some species – Π – of sub-concepts of *P*/collections of *P*s for which Π-hood is *limitlessly* preserved under iteration of the relevant operation.

This notion is, to be sure, relative to one's conception of what constitutes a limitless series of iterations of a given operation. No doubt we start out innocent of any conception of serial limitlessness save the one implicit in one's first idea of the infinite, whereby any countable potential infinity is limitless. Under the aegis of this conception, *natural number* is properly indefinitely extensible with respect to *finite* and so, just as Dummett suggests, indefinitely extensible simpliciter. The crucial conceptual innovation which transcends this initial conception of limitlessness and takes us to the ordinals as classically conceived is to add to the idea that every ordinal has a successor the principle that every infinite series of ordinals has a limit, a first ordinal lying beyond all its elements – the resource encapsulated in Cantor's Second Number Principle. If it is granted that this idea is at least partially – as it were, initial-segmentally – acceptable, the indefinite extensibility of *natural number* will be an immediate casualty of it. (Critics of Dummett who have not been able to see what he is driving at are presumably merely taking for granted the orthodoxy that the Second Number Principle is at least partially acceptable.)

4 Indefinite extensibility: Burali-Forti and Cantor

Very well. Roughly summarised, then, the proposal is that *P* is indefinitely extensible just in case, for some Π, any Π sub-concept of *P* allows of a *limitless* series of Π-preserving enlargements. Since the series of Π-preserving enlargements is limitless, any such concept *P* must indeed allow of an injection of the ordinals into its instances, so Russell's conjecture is confirmed by this account. It is immediately striking, though, that there seems to be nothing immediately paradoxical about indefinite extensibility, so characterised. Why should a concept in good standing not be sufficiently "expansive" to contain a limitlessly expanding series of Π sub-concepts without ever puncturing, as it were? I'll return to this below.

Still, there is a connection with paradox nearby. For example, in case *P* is *ordinal*, and Π*Q* holds just if the *Q*s *exemplify a well-order-type*, it seems irresistible to say that *ordinal* is itself Π. After all, the ordinals are well-ordered. But then the relevant principle of extension, *F*, kicks in and dumps a new object on us that both must and cannot be an ordinal – must because it corresponds, it seems, to a determinate order-type; but cannot because the principle of extension always generates a non-instance of the concept to which it is applied. Thus runs the Burali-Forti paradox.

The question, therefore, is why we have allowed our intuitive concept of *ordinal* to fall, fatally, within the compass of the relevant Π/*F* pair? For *that*, it may seem, is the key *faux pas*. Well, but what option do we have? There is no room for question whether the ordinals are well-ordered. But to be well-ordered is to have an order-type, and we have identified the ordinal numbers with order-types. The only move open, it seems, is to deny that every well-ordered series *is* of a determinate order-type, has an ordinal number. Specifically, it seems we have to deny that *ordinal* itself determines a well-ordered series of a determinate order-type and so has an ordinal number. But the price of that denial is that before we can assure ourselves of the existence of any particular limit ordinal, we need first to know that its putative predecessors are not 'all the ordinals there are'. And this price will be exacted right back at first base, when the issue is that of justifying the existence of ω, the limit of the finite ordinals. In short, the pressure that induces the *faux pas* is just the pressure to allow the ordinals to run into the Cantorian transfinite in a principled fashion in the first place.

The Burali-Forti paradox, and the more general predicament of *ordinal number* that it brings out, thus seems aptly described as indeed exactly a paradox of indefinite extensibility. How close is the comparison provided by *cardinal* number and Cantor's paradox? These remarks of Dummett [1991, pp. 315–316] suggest that he regards the situation as a tight parallel:

> … to someone who has long been used to finite cardinals, and only to [finite cardinals], it seems obvious that there can only be finite cardinals. A cardinal number, for him, is arrived at by counting; and the very definition of an infinite totality is that it is impossible to count it. … [But this] prejudice is one that can be overcome: the beginner can be persuaded that it makes sense, after all, to speak of the number of natural numbers. Once his initial prejudice has been overcome, the next stage is to convince the beginner that there are distinct [infinite] cardinal numbers: not all infinite totalities have as many members as each other. When he has become accustomed to this idea, he is extremely likely to ask, 'How many transfinite cardinals are there?'. How should he be answered? He is very likely to be answered by being told, 'You must not ask that question'. But why should he not? If it was, after all, all right to ask, 'How many numbers are there?', in the sense in which 'number' meant 'finite cardinal', how can it be wrong to ask the same question when 'number' means 'finite or transfinite cardinal'? A mere prohibition leaves the matter a mystery. It gives no help to say that there are some totalities so large that no number can be assigned to them. We can gain some grasp on the idea of a totality too big to be counted … but once we have accepted that totalities too big to be counted may yet have numbers, the idea of one too big even to have a number conveys nothing at all. And merely to say,

> 'If you persist in talking about the number of all cardinal numbers, you will run into contradiction', is to wield the big stick, but not to offer an explanation.

However, I think the parallel is questionable. It is true that we only get the indefinitely extensible series of transfinite cardinals up and running in the first place by first insisting on one-one correspondence between concepts as necessary and sufficient for sameness, and hence existence, of cardinal numbers in general – not just in the finite case – and that the conception of *cardinal number* as embracing both the finite and the spectacular array of transfinite cases thus only arises in the first place when it is taken without question that concepts in general – or at least *sortal* concepts in general: concepts that sustain determinate relations of one-one correspondence – have cardinal numbers, identified and distinguished in the light of those relations. That is how the intuitive barrier to the question, how many natural numbers are there, is overcome. And it is also true that that at least loosens the lid on Pandora's box: for the intuitive barrier to the question, how many *cardinal* numbers are there, is thereby overcome too. But loosening the lid isn't enough to trigger paradox. Hume's principle, identifying the cardinal numbers associated with sortal concepts in general just when those concepts are bijectable, encapsulates exactly the "resistance-overcoming" move that Dummett is talking about. And it generates, indeed, not merely a cardinal number of cardinal numbers but the universal number "Anti-zero", the number of absolutely everything that there is. But it does not spawn any paradox, as far as it goes. It is a consistent principle; at least, it is consistent in classical second-order logic. To get the paradox – Cantor's paradox – out of the notion of cardinal number that Hume's principle characterises, we need to embed it in a set-theory containing the associated principles sufficient to generate Cantor's theorem itself: unrestricted Union, an exemplar set for any given set of cardinals, and a set of all cardinals. None of that baggage is entailed just by the assumption that every sortal concept has a cardinal number, identified and distinguished from others by relations of one-one correspondence.

Moreover, the notion of cardinal number is not needed at all to spring that paradox. Given only a universal set, and unrestricted power set, standard moves in naive set-theory will allow us to prove both that its power set is injectable into the universal set (via a unit set mapping, e.g.) and that there can be no such injection (via the diagonalisation in Cantor's theorem). This is already a paradox. But it is a paradox for the (naïve) notion of *set*. *Cardinal number*, as extended into the transfinite via a criterion of one-one correspondence, is not in play. Someone could reject that extension and still have to confront the antinomy. The core of Cantor's paradox can indeed be assumed under our template for a paradox of indefinite extensibility: simply take P as *object* (or *self-*

identical), ΠQ as *the Qs compose a set* and F as the power-set operation. Consider any such Π concept, Q. The reasoning of Cantor's theorem shows that some of the members of FQ cannot be instances of Q. This immediately gives a contradiction when P itself is taken to be Π, i.e. when we assume a universal set. But no assumptions about *cardinal number* are involved. It is true that, as illustrated earlier, *cardinal number* is indefinitely extensible with respect to *set* when the appropriate assumptions about sets – Union, Power and Replacement – are made, and that this is enough for a paradox of indefinite extensibility if *cardinal number* is itself assumed to determine a set. But this should impress as a frame-up, rather than an insight. The real problem is with the set-theoretic assumptions involved.

Notice, incidentally, that if we deny that *set*, and *cardinal number* themselves determine sets, then we obtain – or at least I know of no reason to doubt that we obtain – examples of the possibility shortly canvassed earlier: concepts that are indefinitely extensible but with whose indefinite extensibility no paradox is associated. The philosophical justifiability of that denial is, naturally, entirely another matter.

Basic Law V

If the foregoing is correct, the cases of two of the 'principal suspects', *ordinal number* and *cardinal number,* are different. The former is unquestionably guilty; the jury should take more time on the latter. When comprehension principles are accepted for the ordinals that both ensure that every well-ordered collection has an ordinal and provides for unlimited applicability of successor and limit, *ordinal number* is essentially both indefinitely extensible and susceptible to a paradox of indefinite extensibility qua satisfying the relevant trigger concept, Π. When comprehension for the cardinals is determined by Hume's principle, it takes set-theoretic assumptions to make a case that *cardinal number* is indefinitely extensible, and further set-theoretic assumptions to make a paradox out of that. These assumptions have no evident intrinsic connection with *cardinal number*.

So what, finally, about *course-of-values* as fixed by Basic Law V? Is it fair, in the light of the account of indefinite extensibility now on the table, and its connection with paradox, to attribute the antinomy that Russell discovered to the indefinite extensibility of the notion that Law V characterises?

I don't think so. There is certainly a paradox of indefinite extensibility in the offing. Here is how it goes. Restrict attention to the case of courses-of-values whose ranges are concepts and whose values truth-values – to the case of *extensions of concepts* – so the axiom becomes, in effect:

$$(\forall P)(\forall Q)(\{x{:}Px\}=\{x{:}Qx\} \leftrightarrow (\forall x)(Px \leftrightarrow Qx))$$

Extensionality and Naïve Comprehension can be read off straightaway: extensions are identical just when their associated concepts are co-extensive; and every concept has one. (Proof: take '*P*' for '*Q*', detach the left-hand-side of the biconditional, and existentially generalise on one occurrence of '{*x*:*Px*}'.) So *absolutely any* concept of extensions is associated with its own extension. Take *P* then as *extension* itself, and Π as *has an extension*. Let *Q* be any subconcept of *P*. By Law V, *Q* has an extension. Define membership in one of the natural ways.[9] Consider the concept *R*: *Qx and not: xεx*. Form the extension of this concept, *r*. Choose this for *FQ*. Suppose *Qr*. Do we have *rεr*? If so then, *r* falls under *R* and is hence a *Q* that is not a member of itself. But *Qr and not: rεr* is in turn is the condition for being a member of *r*. Contradiction. So not *Qr*. Take *Q′* as the concept: *Qx* ∨ *x=r*.

Referring back to the three conditions listed earlier for our initial, relativised notion of indefinite extensibility, the foregoing completes a case for saying that *extension* is indefinitely extensible with respect to *has an extension*. Paradox is then immediate when we reflect that by Basic Law V, Π*P*, i.e. that the Law requires that *extension* itself has an extension (compare: that there is a set of all sets).

But although it exploits a similar trick, that is not quite the paradox that Russell discovered. Paradoxes of indefinite extensibility, as now understood, turn essentially on the application of the principle of extension, *F*, to the concept *P* itself – an application made possible by *P*'s satisfaction of the higher-order trigger concept, Π. That doesn't happen in the reasoning from Basic Law V that Russell found – or a least, that Frege took him to have found. The key resource for that reasoning is simply the license, granted by Basic Law V, to take it that every monadic open sentence with an objectual argument place has an extension, and hence in particular that *x is not self-membered* has an extension. The assumption that *extension* has an extension is not at work in the brewing. It is true that, as with ordinal number, the paradox flows very directly from comprehension principles that go right to the heart of the intended notion. But in the case of Law V, and in contrast to *ordinal number*, the comprehension princi-

9 For instance stipulate that *x* is a member of *y* just if *x* satisfies every *P* of which *y* is the extension.

ple concerned has to do not with an intuitive vision of the desired extent and structure of the relevant population of objects, but simply with the most straightforward view – absolutely integral to Frege's philosophy of mathematics and his treatment of mathematical existence – of the relation between concepts and their associated logical objects.

Dummett's writings on this topic are shot through with the idea that the contradictions are the symptom of a deeper philosophical mistake, that Russell's paradox is, as it were, a carbuncle on the face of an edifice that betrays a deeper underlying malaise. For Dummett, the indefinite extensibility of fundamental mathematical domains is a philosophically vital fact about them, and one gets the impression almost that he regarded the paradox as a fitting nemesis for Frege's failure to understand and acknowledge this fact. But he nowhere says what Frege should have done differently if he had recognised the fact, nor how it would have helped. I am sceptical about the diagnosis, for the reasons I have given. But I do not wish to reject the idea that the contradiction marks a fundamental problem in the way that Frege is thinking about mathematical ontology. There is nothing wrong with the deflationary answer to our leading question that says: Frege fell into paradox because he failed to think through the implications of the full repertoire of open sentences that fall within the range of higher-order quantifiers – failed, if you like, to reckon with the expressive resources, and especially those of diagonalisation, that come with classical, impredicative higher-order logic. He simply "didn't think of that kind of case". But the real question is: why did he adopt an axiom that put his project at risk from that very understandable oversight? And the answer is because the simple correlation postulated by Basic Law V encapsulated a vision of mathematical ontology that was absolutely integral to his logicism: that the mathematical objects of arithmetic and analysis are simply the logical objects that are the Fregean surrogates of functions. That is why the paradox went right to the heart of his philosophy of mathematics, and why his reaction to it was eventually one of despair.

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