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6. Logicism in the Twenty-first Century

Bob Hale
Crispin Wright

1. Frege's Philosophy of Mathematics and the Neo-Fregean Program

Frege believed, at least for most of his career, that the fundamental laws of elementary arithmetic—the theory of the natural numbers (finite cardinals)—and real analysis are *analytic*, in the sense he explained in *Grundlagen* §3—that is, provable on the basis of general logical laws together with suitable definitions. Since his defense of this thesis depended upon taking those theories to concern a realm of independently existing objects (*selbständige Gegenstände*)[1] —the finite cardinal numbers and the real numbers—his view amounted to a *Platonist* version
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of *logicism*. Neo-Fregeanism[2] holds that he was substantially right in both these components of his philosophy, but takes a more optimistic view than Frege himself did of the prospects for contextual explanations of fundamental mathematical concepts—such as those of cardinal number and real number—by means of what are now widely called *abstraction principles*. Generally, these are principles of the form

- $\forall \alpha \forall \beta (\Sigma(\alpha) = \Sigma(\beta) \leftrightarrow \alpha \approx \beta)$,

where Σ is a term-forming operator applicable to expressions of the type of α,β and ≈ is an equivalence relation on entities denoted by expressions of that type. The type of α,β may be that of singular terms, in which case ≈ is a first-level relation and the resulting abstraction principle represents a *first-order* abstraction on *objects*. A well-known example, in terms of which Frege conducted most of his own discussion of this type of explanation, is the Direction Equivalence. [3]

The direction of line *a* = the direction of line *b* if and only if lines *a* and *b* are parallel.

A potentially more important class of *higher-order* abstractions results from taking the type of α and β to be that of first (or higher)-level predicates, with ≈ a correspondingly higher-level relation. One much discussed—and, for our purposes, crucial—principle of this kind is what has come to be known as *Hume's Principle*: the abstraction by means of which, on the proposal reviewed in the central sections (§§60-68) of *Grundlagen*, the concept of (cardinal) number may be explained:

The number of *F*s = the number of *G*s if and only if there is a one-one correlation between the *F*s and the *G*s

As Frege went on to point out (*Grundlagen* §§71-72), what is required for there to be a one-one correlation between the *F*s and the *G*s may itself be explained in purely logical terms. [4]

As is well known, Frege very swiftly came to the conclusion that the explanatory purport of abstraction principles is severely qualified by an objection which had earlier led him, at *Grundlagen* §56, to reject as inadequate an attempt to

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define number by recursive definition of the series of numerically definite quantifiers of the form "there are exactly *n* Fs" As he there puts it:

… we can never—to take a crude example—decide by means of our definitions whether any concept has the number JULIUS CAESAR belonging to it, or whether that same familiar conqueror of Gaul is a number or not. [5]

In his view, the very same difficulty fatally infects the attempt to define *number* by means of Hume's Principle: the proposed definition provides us with a means to decide whether the number of *F*s = *q*, when *q* is given in the form "the number of *G*s"—but it apparently entirely fails to do so, when *q* is not a term of that form (nor a definitional abbreviation of such a term). Perceiving no way to solve the problem, Frege concluded that an altogether different form of definition is needed and opted for his famous explicit definition in terms of extensions:

The number of *F*s = the extension of the concept "equal to the concept *F*"

in effect, that the number of *F*s is the class containing exactly those concepts *G* such that the *G*s are one-one correlated with the *F*s. To derive the fundamental laws of arithmetic with the aid of this definition, Frege obviously required an underlying theory of extensions or classes. This he sought to provide, in *Grundgesetze*, by means of his Basic Law V, which governs what Frege called value ranges of functions (concepts being a particular kind of function, on Frege's account). As regards extensions of concepts, what Basic Law

V asserts is that the extensions of two concepts are identical just in case those concepts have the same objects falling under them, that is (in class notation):

- $\forall F \forall G(\{x : Fx\} = \{x : Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx))$.

As very soon became apparent, this is disastrous, since Basic Law V leads to Russell's Paradox. Frege sought at first to modify his law V so as to avoid the inconsistency, but to no avail, and finally abandoned his attempt to provide a logical foundation for arithmetic and analysis as a complete failure.

Neo-Fregeanism holds that Frege need not have taken the step which led to this unhappy conclusion. At least as far as the theory of natural numbers goes, Frege's central mathematical and philosophical aims may be accomplished by basing the theory on Hume's Principle, adjoined as a supplementary axiom to a suitable formulation of second-order logic. Hume's Principle cannot, to be sure, be taken as a definition in any strict sense —any sense requiring that it provide for the eliminative paraphrase of its definiendum (the numerical operator, "the number of … ") in every admissible type of occurrence. But this does not preclude its being viewed as an *implicit* definition, introducing a sortal concept of cardinal number

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and, accordingly, as being analytic of that concept—and this, the neo-Fregean contends, coupled with the fact that Hume's Principle so conceived requires a prior understanding only of (second-order) logical vocabulary, is enough to sustain an account of the foundations of arithmetic that deserves to be viewed as a form of logicism which, while not quite logicism in the sense of a reduction of arithmetic to logic, or as a demonstration of its analyticity in Frege's own strict sense, [6] preserves the essential core and content of Frege's two fundamental theses.

Restricting attention for the time being to elementary arithmetic, there are two main claims —one logical, the other more purely philosophical—which must be seen to hold good, if the neo-Fregean's leading thesis is to be sustained. The logical claim is that the result of adjoining Hume's Principle to second-order logic is a consistent system which suffices as a foundation for arithmetic, in the sense that all the fundamental laws of arithmetic are derivable within it as theorems. [7] The philosophical claim is that if that is so, that constitutes a vindication of logicism, on a reasonable understanding of that thesis.

In our view, investigations have now reached a point at which at least the technical part of the logical claim may be taken to have been established. [8] Hume's Principle, added to a suitable system of second-order logic, does indeed suffice for a proof of the Dedekind-Peano axioms. [9] Moreover, as far as consistency is concerned, we now have as much assurance as it seems reasonable to demand. In "The Consistency of Frege's *Foundations of Arithmetic*," Boolos (1987) presents a formal theory FA (Fregean Arithmetic),

incorporating an equivalent of Hume's Principle, which captures the mathematical content of the central sections of Frege's *Grundlagen* (§§63-83) and proves not just that we can give a model for Hume's Principle, but also that the informal model-theoretic proof can be replicated both in standard set theory and in the weaker theory known as "second-order
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arithmetic" or "analysis," the consistency of which seems well beyond serious question. The *philosophical* significance of the situation is another matter entirely. If the neo-Fregean is to justify his contention that Frege's Theorem can underwrite a viable Platonistic version of logicism, then, even if he restricts that claim to elementary number theory, he has much philosophical work to do on several fronts; and if he additionally aspires, as we do, to sustain a more inclusive version of logicism encompassing at least the theory of real numbers and perhaps some set theory, then he must take on an additional range of tasks—some technical, some philosophical.
In the four immediately following sections, we shall confine ourselves to issues which neo-Fregeanism must address, even if the scope of its leading claims is restricted to elementary arithmetic. Many of these concern the capacity of *abstraction principles*—centrally, but not only, Hume's Principle itself—to discharge the implicitly definitional role in which the neo-Fregean casts them, and thereby to subserve a satisfactory apriorist epistemology for (at least part of) mathematics. Others, to be briefly reviewed in section 8, concern the other main assumption that undergirds the specifically logicist aspect of the neo-Fregean project (and equally, of course, Frege's original project): that the logic to which abstraction principles are to be adjoined may legitimately be taken to include *higher-order*—at the very least, second-order—logic without compromise of the epistemological purposes of the project. In between, in sections 6 and 7 respectively, we shall canvass some of the issues that attend a neo-Fregean construction of analysis and of set theory.

2. Abstraction Principles—Ontology and Epistemology

Frege's Platonism is the thesis that number words have reference, and that their reference is to objects—objects which, on any reasonable account of the abstract-concrete distinction, [10] must be reckoned to lie on the abstract side of it. Why it was so crucial for Frege that numbers be recognized as objects becomes clear when we consider how he proposes to prove that every finite number is immediately followed by another (and hence, in the presence of others of the Dedekind-Peano axioms, which Frege can straightforwardly establish, that the sequence of finite numbers is infinite). The idea of Frege's proof is to show that for each finite *n*, the end p.170

number belonging to the concept *finite number ancestrally preceding or equal to n* itself directly follows after *n*. To prove this, in the manner sketched in *Grundlagen* §§82-83, requires applying Frege's criterion of identity for cardinal numbers, as encoded in Hume's Principle, to first-level concepts of the type just indicated—that is, concepts under which numbers, and indeed only numbers, fall—and thus requires numbers to be objects.

It may be rejoined that this gives Frege a *motive* to treat numbers as objects, but not a *justification* for doing so. However, while *Grundlagen* provides no systematic argument for this apparently crucial thesis, Frege did at least provide hints, and some of the materials, on the basis of which a case for it may be constructed. We start from two ideas. First: *objects*, as distinct from entities of other types (properties, relations, or, more generally, functions of different types and levels), just are what (actual or possible) singular terms refer to. Second: no more is to be required in order for there to be an at least prima facie case that a class of apparent singular terms have reference, than that they occur in certain true statements free of all epistemic, modal, quotational, and other forms of vocabulary standardly taken to compromise straightforward referential function. In particular, if certain expressions function as singular terms in various true first-order atomic contexts, there can be no further question that they have reference and, since they are singular terms, refer to objects. The underlying thought is that—from a semantic point of view—a singular term just *is* an expression whose function is to effect reference to an object, and that a wide class of statements containing such terms cannot be true unless those terms successfully discharge their referential function. Provided, then (as certainly appears to be the case), there are such true, suitable statements so featuring numerical singular terms, there are objects—numbers—to which they make reference. [11]

This simple argument, of course, is not conclusive. At most it creates a presumption in favor of numbers' existence as a species of objects. And there are various ways in which that presumption might be defeated. An opponent might agree that the statements to which appeal is made have, superficially, the right logical form, but deny that appearances are trustworthy. Or, more radically, she

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may accept those appearances as a fair guide to truth-conditions (at least as far as purely arithmetical statements go), but deny that any of the relevant statements are actually—taken literally and at face value—*true*.[12] It might be conceded that the argument makes a prima facie case, but argued that the relevant statements cannot really be seen as involving reference to numbers, on the ground that there are (allegedly) insuperable obstacles in the way of making sense of the very idea that we are able to engage in identifying reference to, or thought about, any (abstract) objects to which we stand in no spatial, causal or other natural relations, however remote or indirect. A closely related objection—originating in Paul Benacerraf's much discussed dilemma for accounts of mathematical truth, and subsequently pressed in a revised form by Hartry Field—has it that a Platonist account of the truth-conditions of mathematical statements puts them beyond the reach of humanly possible knowledge or reliable belief, and must therefore be rejected.

The simple argument—it is worth noting—makes no appeal to the specific possibility of explaining singular terms for natural numbers, along with the corresponding sortal concept, via Hume's Principle. Of course, the case for the existence of numbers *can* be made on the basis of Hume's Principle, and it is important to the neo-Fregean that this should be so, precisely because it provides for a head-on response to the epistemological challenge posed by Benacerraf's dilemma. Hume's Principle, taken as implicitly defining the numerical operator, fixes the truth-conditions of identity-statements featuring canonical terms for numbers as those of corresponding statements asserting the existence of one-one correlations between appropriate concepts. Given statements of the latter sort as premises, the truth of such identities (and hence the existence of numbers) may accordingly be inferred. We thus have the makings of an epistemologically unproblematic route to the existence of numbers and a fundamental species of facts about them. The key idea is clearly present in Frege's metaphorical explanation of how the Direction Equivalence may serve to introduce the concept of direction:

"we carve up the content [expressed in the statement that lines *a* and *b* are parallel] in a new way, and this yields us a new concept" [i.e., of direction]. (Frege 1884, §64) More generally, it is open to us, by laying down an abstraction principle $\forall\alpha\forall\beta(\Sigma(\alpha) = \Sigma(\beta)\leftrightarrow\alpha\approx\beta)$, to re-describe or re-conceptualise the type of state of affairs apt to be depicted by sentences of the form $\alpha \approx \beta$—we may so reconceive such states of affairs that they come to constitute the identity of a new kind of object of which, by that very stipulation, we introduce the concept. [13]

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This is all quite consistent with Platonism, modestly and soberly conceived. For it is no part of the proposal that objects of the new kind—abstract objects such as directions, numbers, or whatever—are creations of the human mind, somehow brought into being by our stipulation. The sense in which they are new is merely that the (sortal) concept under which they fall is newly introduced by our abstractive explanation. In and of itself, the abstraction does no more than introduce that concept and establish a use for a corresponding range of singular terms by means of which its instances, if any, may be designated, and involves no attempt to guarantee that the concept does indeed have instances. The existence of objects of the new kind (e.g., directions) depends—and, if the explanation is accepted, depends exclusively—upon whether or not the relevant

equivalence relation (e.g., parallelism) holds among the entities of the presupposed kind (e.g., lines) on which that relation is defined. [14] And, to return to the case that most concerns us here, provided that facts about one-one correlation of concepts—in the basic case, sortal concepts under which only concrete objects fall—are, as we may reasonably presume, unproblematically accessible, we gain access, via Hume's Principle and without any need to postulate any mysterious extrasensory faculties or so-called mathematical intuition, to corresponding truths whose formulation involves reference to numbers. [15]
Both this form of the case for the existence of numbers and its claim to provide trouble-free access to a basic class of facts about them may, like the simple argument, be challenged. For example, it may be claimed that if statements having the form of the left-hand side of Hume's Principle involve reference to numbers, then they *cannot* legitimately be accepted as equivalent to corresponding statements having the form of the right-hand side, since the latter involve no such reference and may accordingly be true under circumstances where their left-hand counterparts are not (viz. if there are no numbers). On this basis, it may be held that Hume's Principle—along with abstraction principles quite generally—is acceptable only *either* if it is so construed that the apparent reference to numbers presented by the left-hand sides of its instances is treated as merely apparent, *or* it is amended so as to explicitly allow for the presupposition that numbers exist.

An advocate of the first alternative—likely to find favor with orthodox nominalists—will insist that we may accept abstraction principles as explanatory of the truth conditions of their left-hand sides only if we treat the latter as devoid of semantically significant syntax, beyond the occurrences of the term or predicate variables α and β. The only legitimate reading of Hume's Principle, on this view, is an *austere* one according to which it merely serves to introduce a new, semantically unstructured two-place predicate:
the number of … is identical with the number of__
as alternative notation for the equivalence relation that figures on the right-hand side: " … corresponds one-one with __." But why insist upon austerity? The seemingly straightforward answer—that since the statement "F corresponds one-one with G" plainly involves no reference to numbers, it can be taken as explaining the truth conditions of "the number of Fs = the number of Gs" only if the latter likewise involves no such reference—implicitly relies upon a question-begging assumption. It is certainly true that statements of one-one correspondence between concepts involve no terms purporting reference to numbers. And it is equally clearly true that two statements cannot have the same truth-conditions if they differ in point of existential commitment. But the premise the nominalist requires—if he is to be justified in inferring that statements of one-one correspondence can be equivalent to statements of numerical identity only if the latter are austerely construed—is that statements of one-one correspondence do not demand the existence of numbers.

And this premise does not follow from the acknowledged absence, in such statements, of any explicit reference to numbers. The statement that a man is an uncle involves no explicit reference to his siblings or their offspring, but it cannot be true unless he has a nonchildless brother or sister.

It may be objected that while no one would count as understanding the statement that Edward is an uncle if she could not be brought to agree that its truth requires the existence of someone who is brother or sister to Edward and father or mother of someone else, it is quite otherwise with statements of one-one correspondence. Someone may perfectly well understand the statement that the Fs correspond one-one with the Gs without being ready to acknowledge—what the nominalist denies—that its truth requires the existence of numbers. The neo-Fregean reply is that so much is perfectly correct, but insufficient for the nominalist's argument. It is *correct* precisely because—just as the neo-Fregean requires—understanding talk of one-one correspondence between concepts does not demand possession of the concept of *number*. But it is *insufficient* because the question that matters is, rather, whether one who is fully cognizant of *all* the relevant concepts—that is, the concepts of *one-one correspondence between concepts* and *number*—could count as fully understanding a statement of one-to-one correspondence between concepts without being ready to agree that its truth

called for the existence of numbers. If the concept of *number* is implicitly defined, as the neo-Fregean proposes, by Hume's Principle, she could not. Insisting at this point that no such explanation can be admitted, because only an austere reading of Hume's Principle is permissible, amounts to no more than an unargued—and now explicitly question-begging—refusal to entertain the kind of explanation on offer.

Nominalists of the less traditional kind represented, most prominently, by Hartry Field[16] are more likely to favor the second alternative: to allow that abstraction principles are all very fine as stipulative explanations, provided (what the nominalist views as) their existential presuppositions are made properly and fully explicit. Hume's Principle, in particular, should be replaced by a properly *conditionalized* version along the lines of "If there exist such things as the number of *F*s and the number of *G*s, then they are identical if and only if the *F*s and *G*s are one-one correlated." Such conditionalized principles would, of course, be inadequate to the neo-Fregean's purposes.

There is, however, an immediate difficulty confronting the proposal to replace Hume's Principle by such a conditionalized version. [17] How are we to understand the antecedent condition? In rejecting Hume's Principle as known a priori, Field holds that the obtaining of a one-one correlation between a pair of concepts cannot be regarded as *tout court* sufficient for the truth of the corresponding numerical identity. So, since that was an integral part of the proposed implicit definition of the concept of number, his position begs some other account of that concept—an intelligible doubt about the existence of numbers requires a concept in terms of which the doubt might be framed. More specifically, if we take it that the hypothesis that numbers exist may be rendered as "$\exists F \exists x\, x = \mathrm{N}y\mathrm{F}y$," then in order to understand the condition under which Field is prepared to allow that Hume's Principle holds, we must *already* understand the numerical operator. But it was the stipulation—unconditionally—of Hume's Principle which was supposed to explain it. That explanation has lapsed; but Field has put nothing else in its place.

Is there perhaps a better way to implement Field's conditionalization strategy, one which avoids this difficulty? A familiar way of thinking about the manner in which theoretical scientific terms acquire meaning[18] holds that we should view a scientific theory, embedding one or more novel theoretical terms, as comprising two components: one encapsulating the distinctive empirical content of the theory without deployment of the novel theoretical vocabulary, the other serving to fix the

meaning(s) of the theoretical term(s). The theory's total empirically falsifiable content is, roughly, that there exist entities of a certain kind, namely, entities satisfying (a schematic formulation of) the (basic) claims of the theory. This can be expressed by the theory's *Ramsey sentence* (i.e., roughly, an existential generalization obtained from the original formulation of the theory employing the new theoretical terms by replacing each occurrence of each new term with a distinct free variable of appropriate type, and closing the resulting open sentence by prefixing the requisite number of existential quantifiers). Thus if, focusing for simplicity on the case where a single new theoretical term, "f," is introduced, the undifferentiated formulation of the theory is "$\Theta(f)$", then its empirical content is exhaustively captured by its Ramsey sentence, "$\exists x\, \Theta(x)$," where the new variable "x" replaces "f" throughout "$\Theta(f)$." The new term "f" can then be introduced, by means of what is sometimes called the *Carnap conditional*[19] : "$\exists x\, \Theta(x) \rightarrow \Theta(f)$", as denoting whatever (if anything) satisfies "Θ" (on the intended interpretation of the old vocabulary from which it is constructed). This conditional expresses, in effect, a convention for the use of the new term "f." Being wholly void of empirical content, it *can* be stipulated, or held true a priori, without prejudice to the empirical disconfirmability of the theory proper. Alternatively, the definitional import of the theory might be seen as carried, rather, by a kind of *inverse* of its Carnap conditional—in the simple instance under consideration, something of the shape "$\forall x(x = f \rightarrow \Theta(x))$." We shall follow this proposal in the brief remarks to follow. [20]

This may seem to offer Field a way around the difficulty: treat the system consisting of Hume's Principle and second-order logic as a "theory," in a sense inviting comparison with empirical scientific theories, whose capacity to introduce theoretical concepts by implicit definition is uncompromised by the fact that they may turn out to be false. We may indeed think of this theory as *indirectly* implicitly defining the concept of cardinal number; however, the real vehicle of this definition is not Hume's Principle but the corresponding inverse-Carnap conditional:

- (HP*) $\forall F \forall G \forall u \forall v((u = \mathrm{N}x\mathrm{F}x \wedge v = \mathrm{N}x\mathrm{G}x) \rightarrow (u = v \leftrightarrow F\text{1-1}G))$.

And now the complaint that Field has put nothing in place of Hume's Principle to enable us to construe the condition on which he regards it as a priori legitimate to affirm Hume's Principle is met head-on. HP*, Field may say, tells us what
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numbers are in just the way that the inverse-Carnap conditional for any (other) scientific theory tells us what the theoretical entities it distinctively postulates are—by saying what (fundamental) law(s) they characteristically satisfy, *if* they exist. That there are numbers is itself no conceptual or definitional truth; it is, rather, the content of a theory (in essence, a theory given by the Ramsified version of Hume's Principle:

$\exists\eta \forall F \forall G(\eta F = \eta G \equiv F \text{ 1-1 } G)$), which may perfectly well be—and in Field's view is—false. [21]

Should this proposal be accepted? Let's allow that *if* there is good reason to insist that an implicit definition of the numerical operator should proceed, not through an outright stipulation of Hume's Principle but through something more tentative, then one plausible shape for the stipulation is an inverse Carnap formulation of the kind suggested. But is there any such reason?
The comparison with the empirical scientific case cannot provide one. Conditionalization is called for in the scientific case in order to keep open the possibility of empirical disconfirmation. Fixing the meaning of "f" by stipulating the truth of the whole conditional '$\forall x(x = f \rightarrow \Theta(x))$' leaves room for acknowledgment that the antecedent (more precisely, its existential generalization '$\exists x\, x = f$') might turn out false—grounds to think it false being provided by empirical disconfirmation of the consequent (more precisely, of the theory's Ramsey-sentence '$\exists x \Theta(x)$'). But there is no possibility of empirical disconfirmation of the Ramsified Hume's Principle if, as seems reasonable to suppose, Hume's Principle is *conservative* over empirical theory—as Field allows, indeed argues—since in that case it will have no proper empirical consequences.
It does not, of course, follow that there cannot be other reasons to insist upon a more cautious, conditionalized form of stipulation. Let us say that a stipulation of a sentence as true is *arrogant* if its truth requires that some of its ingredient terms have reference in a way that cannot be guaranteed just by providing them with a sense. An example would be the stipulation 'Jack the Ripper = the perpetrator of the Whitechapel murders' offered by

way of introduction of the term 'Jack the Ripper.' Were the outright stipulation of Hume's Principle rightly regarded as arrogant, that would—plausibly—provide an independent reason for insisting that it cannot be known a priori just on the basis of its meaning-conferring credentials and for adopting something like the distinction between

factual content, carried by Ramsified Hume, and meaning-fixing—accomplished by something appropriately conditionalized—proposed in the scientific-theoretical case. The crucial question is, then, whether an outright stipulation of Hume's Principle is indeed guilty of this failing. We think not. In general, it would seem, it suffices for an implicitly definitional stipulation to avoid arrogance that it does no more than lay down introduction and elimination conditions for the definiendum in terms themselves free of any problematic existential presupposition. And this will surely be so, if the import of the stipulation may be parsed into introductory and/or eliminative components, each conditional in form, prescribing which true statements free of occurrences of the definiendum are to be respectively sufficient and/or necessary for true statements variously embedding it.

But that is exactly what Hume's Principle, proposed as a stipulation, does. The principle does not just assert the existence of numbers as 'Jack the Ripper is the perpetrator of this series of killings' asserts the existence of the Ripper. What it does is to fix the truth conditions of identities involving canonical numerical terms as those of corresponding statements of one-one correlation among concepts (compare the schematic stipulation '*a* is the single perpetrator of these killings if and only if *a* is Jack the Ripper'). Its effect is that one kind of context free of the definiendum—a statement of one-one correlation between suitable concepts—is stipulated as sufficient for the truth of one kind of context embedding the definiendum: that identifying the numbers belonging to those respective concepts. That is its introductory component. And, conversely, the latter type of context is stipulated as sufficient for the former. That is the principle's eliminative component. All thus seems squarely in keeping with the constraint that in order to avoid arrogance, legitimate implicit definitions must have an essentially conditional character. [22] If the additional conditionalization in HP* is proposed in the interests of avoiding arrogance, it thus merely involves a condition too many. [23]

Obviously the general notion of implicit definition invoked here calls for further clarification and defense. [24] But even if it is accepted that this is one way in which a statement might qualify as analytic in something akin to the general spirit of Frege's notion, if not its detail, there are several more specific grounds on which it might be denied that Hume's Principle in particular can function as such an implicit definition. In the next three sections we summarize the most important of them, with just the briefest indication how the neo-Fregean may respond.

3. Abstraction Principles and Julius Caesar

One paramount such ground is, of course, the Julius Caesar problem. The concept of *number* to be explained, in both Frege's and the neo-Fregean view, is a *sortal* concept. Mastery of any general concept applicable to objects involves knowing what distinguishes objects to which it applies from those to which it does not—involves, that is, a grasp of what Dummett has called a *criterion of application* for the concept. What sets sortal concepts apart from others is that full competence with a sortal concept involves, in addition, knowing what settles questions of identity and distinctness among its instances—what determines, given that *x* and *y* are both *F*s, whether they are one and the same, or distinct, *F*s. In other words, sortal concepts are distinguished from others by their association with what Frege called a *criterion of identity*.[25] Hume's Principle appears, at least, to take care of a necessary condition for *number* to be a sortal concept in this sense—by supplying a criterion of identity—and so to constitute at least a partial explanation of a sortal concept of number. The neo-Fregean, however, makes a stronger claim—that by stipulating that the number of *F*s is the same as the number of *G*s just in case the *F*s are one-one correlated with the *G*s, we can set up *number* as a sortal concept (i.e., that Hume's Principle *suffices* to explain the concept of *number* as a sortal concept). This stronger claim appears open to an obvious objection: simply, that Hume's principle cannot be sufficient, since there is also required, as with any other concept, a criterion of application. Since such a criterion would determine which sorts of things are *not* numbers, it would also settle the truth-conditions of any identity statement of the form "the number of *F*s = *q*," where "*q*" marks a place for a term explicitly purporting to denote a thing of some previously understood sort. In short, the Caesar Problem, in the form in which Frege expresses it at *Grundlagen* §66, is nothing other than the problem of supplying a criterion of application for *number* and thereby of setting it up as the concept of a genuine sort of object. But that it is such a concept is the cardinal thesis of (neo-) Fregean Platonism.
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For this reason, [26] we believe it is no option for the neo-Fregean to declare "mixed" identity statements of the troublesome sort out of order or somehow ill-formed. Rather, he must grant that they raise perfectly genuine, legitimate questions and seek to explain how, within the resources at his disposal, their truth-values may in principle be resolved. In short, he needs a positive solution to the Caesar Problem. The best hope for providing one remains, in our view, the consideration that there are at least some restrictions on the extension of any given sortal concept which are mandated by the type of canonical ground —criterion of identity—associated with identity statements concerning its instances; and, more specifically, that because they are canonically decided by reference to independent considerations, statements of numerical and of, say, personal identity must be reckoned to concern different categories (or ultimate sorts) of objects. [27]

4. Abstraction Principles and Bad Company

Frege's Basic Law V, like Hume's Principle, is a second-order abstraction, and thus provides a sharp reminder that not every imaginable abstraction principle constitutes an acceptable means of introducing a concept. Some constraints are needed—with consistency, obviously, a minimum such constraint. But some critics have thought the inconsistency reveals far more—not merely that Law V cannot function as an explanation of a coherent concept of extension or set, but also that abstractive explanations are defective quite generally. If it were possible satisfactorily to explain the concept of *number* by laying down Hume's Principle, the objection goes, then it ought to be possible to do the same for the concept of *extension* by laying down Basic Law V. Since the concept of *extension* cannot be satisfactorily so explained, neither can the concept of *number*, and nor can any other concept.

This is the Bad Company objection in its simplest, most sweeping, and—in our estimation —least challenging form. The objection as stated seems to assume that if a pattern of concept explanation—in this case, explanation by means of an abstraction principle—can be exploited to set up the resources for derivation of

an antinomy, that somehow shows not just that that particular attempt at explaining a concept aborts, but that the pattern itself is defective. This extreme claim should be rejected. Allowed a sufficiently free hand in explaining predicates by laying down their satisfaction conditions, we can explain a predicate applicable to predicates by stipulating that a predicate satisfies "*x* is heterological" if and only if it does not apply to itself, and speedily arrive at the well-known contradiction that "heterological" is heterological just in case it is not. No one seriously supposes that the moral in that case is that we can *never* satisfactorily explain a predicate by saying what it takes for an object to satisfy it. Why suppose it any less unreasonable to draw the parallel moral in the case in hand—that one can never explain a sortal concept by means of a Fregean abstraction? If it should be demanded that the neo-Fregean point to some relevant difference between good and bad cases of would-be abstractive explanations, it would seem, so far, a perfectly adequate reply that Basic Law V, for example, is known to be inconsistent, whereas Hume's Principle is not. Indeed, he can in this case make a stronger reply—although it is not clear that he needs to make it, in order to dismiss the objection as stated—that Hume's Principle is very reasonably believed *not* to be inconsistent.

A potentially more interesting form of Bad Company objection, due to George Boolos, [28] allows that the possibility of inconsistent abstractions does not discredit abstractive explanation as such, and grants that an explanation based on Hume's Principle may justifiably be presumed at least consistent, but points to the possibility of other equally consistent abstraction principles which are, however, inconsistent with Hume's Principle. Hume's Principle has models, but none which have less than countably infinite domains. By taking a different equivalence relation on (first-level) concepts, such as one which holds between a pair of concepts *F* and *G* if and only if just finitely many objects are either *F*-but-not-*G* or *G*-but-not-*F*, we can frame an abstraction—the *Nuisance Principle*[29] —which has it that certain objects ν(*F*) and ν(*G*)—the "nuisances" of *F* and *G*, respectively—are one and the same just in case *F* and *G* are so related. The Nuisance Principle is, like Hume's Principle, consistent. The trouble is that while it has models, it can be shown to have only *finite* models. What this threatens is not so much the capacity of Hume's Principle to function as an explanation, as its title to be regarded as known in virtue of being analytic of the concept of *number*. For does not the Nuisance Principle enjoy an equally good title to be regarded as analytic of the concept of *nuisance*? Yet inconsistent principles cannot both be

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true, much less known. Pending exposure of some relevant disparity, should we not conclude that *neither* principle can, after all, be known on the basis of its stipulation?

If the neo-Fregean view of the philosophical significance of Frege's Theorem is to be sustainable, there must *be* a relevant difference: some further constraint(s), in addition to any which may be needed to ensure satisfaction of the minimal (and of course fundamental) condition of consistency, with which good abstractions must comply, but which the Nuisance Principle breaches. One natural proposal is to the effect that no acceptable explanation—whether it proceeds through an abstraction principle or takes some other shape—can carry implications for the size of extensions of concepts other than, and quite unconnected to, the concept to be explained. The requirement this suggests is that an acceptable abstraction principle should be *conservative* in a sense closely akin to that deployed by Hartry Field in his defense of Nominalism, according to which a mathematical theory M is conservative if its adjunction to a nominalistically acceptable theory N has no consequences for the ontology of N which are not already consequences of N alone. [30]

Since the Nuisance Principle constrains the extensions of all other concepts to be at most finite, it fails this requirement and cannot, therefore, constitute an acceptable explanation. Evidently the same will go for any other "limitative" abstraction principle (i.e., any abstraction which, while consistent, can have no models exceeding some assignable cardinality. [31]

There is a wider question here which merits investigation in its own right, quite apart from its obvious bearing on the prospects for extending the neo-Fregean approach beyond elementary arithmetic. What conditions, in general, are necessary and sufficient for an abstraction principle to be acceptable? The apparent need for conservativeness constraints, over and above any restrictions which may be needed to avoid inconsistency, is one important aspect of this larger question. Another issue concerns the requirement of consistency itself. If an appreciable body of mathematics can be generated from acceptable abstraction principles, we should not expect to be able to formulate a general theory of abstraction that is *provably* consistent—save, perhaps, relative to the assumption

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of the consistency of some yet more powerful theory. It may, nevertheless, be possible to articulate some general restrictions which must be observed if inconsistency is to be avoided. One obvious danger here arises from the fact that an equivalence relation defined on the concepts on a specified underlying domain of objects may partition those concepts into more equivalence classes than there are objects in the underlying domain, so that a second-order abstraction may "generate" a domain of abstracts strictly larger than the initial domain of objects. This, in itself, need be no bad thing—indeed, it is essential, if there is to be a neo-Fregean abstractionist route to (classical) analysis. But it raises the specter of something like Cantor's Paradox. If that is to be avoided, it is natural to think, then we must bar all "inflationary" abstractions—abstractions which make impossible demands upon the cardinality of any determinate objectual domain. [32] But how precisely such a restriction should be formulated is a delicate and difficult question which we cannot further pursue here. [33]

There are several other grounds on which the analyticity of Hume's Principle may be, and indeed has been, disputed, to which we can here merely draw the reader's attention. Most obviously, perhaps, it may be claimed that no statement encumbered with significant existential implications can possibly be analytic, in any reasonable sense. Hume's Principle, however, certainly does carry such implications, since it entails the existence of infinitely many objects—each and every one of the natural numbers, at least. Again, while we know that Fregean arithmetic is consistent if (and only if) second-order arithmetic is consistent, we have no absolute guarantee that either theory is consistent; but if we don't know that Fregean arithmetic is consistent, how can we justifiably claim that Hume's Principle is analytic? Or—focusing again upon the principle's existential implications—it may be contended that, quite apart from the fact that no analytic principle ought to carry existential commitments at all, Hume's Principle carries particular such commitments which are, at best, highly questionable. For given that the principle, in conjunction with the second-order logical truth that there is a one-one map from any empty concept to itself, ensures the existence of zero defined as N*x*:*x* ≠ *x*, it would seem that the neo-Fregean can have no ground for refusing to coinstantiate its second-order variables *F* and *G* by the complementary predicate of *self-identity*, so that the principle entails (together with another obvious second-order logical truth) the existence of the universal number N*x*:*x* = *x*—the number, "anti-zero," of all the things that there are. But the existence of such a number is doubtful at best—perhaps worse, since under standard definitions, it is provable in

Zermelo-Fränkel set theory (ZFC) that there can be no cardinal number of all the sets there are. [34]

5. Abstraction Principles and Impredicativity

Formulated in the language of second-order logic, and with quantifiers made explicit, Hume's Principle is

- $\forall F \forall G[Nx : Fx = Nx : Gx] \leftrightarrow \exists R \forall x(Fx \rightarrow \exists !y(Gy \wedge Rxy)) \wedge (Gx \rightarrow \exists !y(Fy \wedge Ryx)))$.

If we are to be able to prove, by means of this principle, the distinctness of the individual natural numbers as Frege proposed to define them ($0 =_{df} Nx{:}x \neq x$, $1 =_{df} Ny{:}(y = Nx{:}x \neq x)$, etc.), and to prove there to be infinitely many of them by proving, as Frege envisaged, that each natural number *n* is immediately followed by the number of numbers up to and including *n*, then it is essential that the principle's initial second-order quantifiers be taken as ranging over concepts under which numbers themselves fall, and hence that its first-order quantifiers be taken as ranging over numbers as well as objects of other kinds. In short, if Hume's Principle is effectively to discharge the role in which the neo-Fregean casts it, its first-order quantifiers must be construed as ranging over, inter alia, objects of that very kind whose concept it is intended to introduce. Is that a cause for concern?
That it is—that the unavoidable impredicativity of Hume's Principle amounts to some sort of vicious *circularity*—has long been the general thrust of much of Michael Dummett's objection both to Frege's philosophy of arithmetic and, more recently, to neo-Fregean efforts to resuscitate it. Indeed, while Dummett has

developed a number of quite distinct lines of criticism of that enterprise, [35] it seems clear that it is the impredicativity of Hume's Principle which forms the focus of his principal complaint. It is, however, neither obvious that impredicativity is always harmful, nor easy to be clear why, exactly, Dummett takes it to be a fatal defect in this case. There is, of course, the comparison with Basic Law V—the form of Bad Company objection noted earlier. It is true enough that both principles are impredicative, and true also that the derivation of Russell's contradiction exploits precisely this feature of Law V—predicativity restrictions would certainly block the derivation, and in that sense, impredicativity may be held to be at least partly to blame for the contradiction. But impredicativity does not always lead to contradiction, and Hume's Principle is, as we have emphasized, at least very plausibly taken to be consistent. Dummett's thought seems to be that impredicativity is objectionable, regardless of whether it actually results in inconsistency. But why?

There seem to be a number of distinguishable lines of objection suggested by Dummett's discussion. One is that, for Frege at least, it was necessary to ensure that every statement about numbers possesses a determinate truth-value, and that the impredicativity of Hume's Principle obstructs this. [36] Another is that quantification over a domain of objects is in good order only if it is possible to supply an independent, prior characterization of the domain, and that this cannot be done for the domain over which Hume's Principle's impredicative first-order quantifiers are required to range. [37] Finally, there is the thought that the impredicative character of Hume's Principle disqualifies it from functioning as an explanation of the numerical operator because, considered as such, it would be viciously circular. [38] None of these objections, in our view, carries conviction. The first appears to rely either upon the questionable assumption that Frege's unswerving commitment to bivalence is indispensable to anything worth describing as a version of logicism or upon an unjustifiably exacting interpretation of the requirements for contextual explanation of a sortal concept. The second invokes an exorbitant—and indeed incoherent—condition on the intelligibility of quantification. And granted—crucially—the availability of a positive solution to the Caesar Problem, it does seem possible, contra the third, to explain how intelligent reception of Hume's Principle as an explanation could enable one, impredicativity notwithstanding, to understand numerical statements of arbitrary complexity. [39]

6. Neo-Fregean Real Analysis

The minimal formal prerequisite for a successful neo-Fregean foundation for a mathematical theory is to devise presumptively consistent abstraction principles strong enough to ensure the existence of a range of objects having the structure of the objects of the intended theory—in the case of elementary arithmetic, for instance, the existence of a series of objects having the structure of the natural numbers (i.e., constituting an ω sequence). Second-order logic, augmented by the single abstraction, Hume's Principle, accomplishes this formal prerequisite. The outstanding question is then whether Hume's Principle, beyond being presumptively consistent, may be regarded as acceptable in a fuller, *philosophically interesting* sense. That raises the intriguing complex of metaphysical and epistemological issues just reviewed.

In parallel, the minimal formal prerequisite for a successful neo-Fregean foundation of real analysis must be to find presumptively consistent abstraction principles which, again in conjunction with a suitable—presumably second-order—logic, suffice for the existence of an array of objects that collectively comport themselves like the classical real numbers; that is, compose a complete, ordered field. Recently a number of ways have emerged for achieving this result.

One attractive approach is known as the *Dedekindian Way*.[40] We start with Hume's Principle plus second-order logic. Then we use the *Pairs* abstraction,

- $(\forall x)(\forall y)(\forall z)(\forall w)(\langle x, y\rangle = \langle z, w\rangle \leftrightarrow x = z \wedge y = w),$

to arrive at the ordered pairs of the finite cardinals so provided. [41] Next we abstract over the *Differences* between such pairs,

- $\mathrm{Diff}(\langle x, y\rangle) = \mathrm{Diff}(\langle z, w\rangle) \leftrightarrow x + w = y + z,$

and proceed to identify the *integers* with these differences. We then define addition and multiplication on the integers so identified and, where *m*, *n*, *p*, and *q* are any integers, form *Quotients* of pairs of integers in accordance with this abstraction:

- $Q\langle m, n\rangle = Q\langle p, q\rangle \leftrightarrow (n = 0 \wedge q = 0) \vee (n \neq 0 \wedge q \neq 0 \wedge m \times q = n \times p).$

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We may now identify a *rational* with any quotient Q $\langle m,n\rangle$ whose second term *n* is nonzero. Then, defining addition and multiplication and the natural linear order on the rationals so generated, we can move on to the objects which are to compose the sought-for completely ordered field via the Dedekind-inspired *Cut Abstraction*:

- $(\forall P)(\forall Q)(\mathrm{Cut}(P) = \mathrm{Cut}(Q) \leftrightarrow (\forall r)(P \leq r \leftrightarrow Q \leq r),$

where '*r*' ranges over rationals and the relation ≠ holds between a property, *P*, of rationals and a specific rational number, *r*, just in case any instance of *P* is less than or equal to *r* under the constructed linear order on the rationals. Cuts are the same, accordingly, just in case their associated properties have exactly the same rational upper bounds. Finally, we identify the *real numbers* with the cuts of those properties *P* which are both bounded above and instantiated in the rationals.
On the Dedekindian Way, then, successive abstractions take us from one-to-one correspondence on concepts to cardinals, from cardinals to pairs of cardinals, from pairs of finite cardinals to integers, from pairs of integers to rationals, and finally from concepts of rationals to (what are then identified as) reals. Although the path is quite complex in detail and the proof that it indeed succeeds in the construction of a completely ordered field is at least as untrivial as Frege's Theorem, it does make for a near-perfect neo-Fregean capture of the Dedekindian conception of a real number as the cut of an upper-bounded, nonempty set of rationals. True, the abstractions involved do not provide for the transformability of any statement about the reals, so introduced, back into the vocabulary of pure second-order logic with which we started out. But that—*pure* logicist—desideratum was already compromised in the construction of elementary arithmetic on the basis of Hume's Principle. As in that case, a weaker but still interesting version of logicism remains in prospect. If each of the successive abstractions invoked on the Dedekindian Way succeeds as an implicit definition of the truth-conditions of contexts of the type schematized on its left-hand side, then there is a route of successive concept formations that starts within second-order logic and winds up with an understanding of the Cuts and a demonstration of the fundamental laws of a canonical mathematical theory of them: in effect, a foundation for analysis in second-order logic and (implicit) definitions. Dedekind did not have the notion of an abstraction principle. But it seems likely that his logicist sympathies would have applauded this construction and its philosophical potential. [42]
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The Dedekindian Way contrasts significantly with another route explored by one of the present authors in recent work. [43] In claiming to supply a foundation for analysis—in particular, in claiming that the series of abstractions involved effectively leads to the real numbers—the Dedekindian Way may be viewed as resting on an essentially *structural* conception of what a real number is: in effect, the idea of a real number merely as a location in a certain kind of (completely) ordered series. For one following the Dedekindian Way, success just consists in the construction of a field of objects—the Cuts, as defined—having the structure of the classical continuum. Against that, we may contrast what is accomplished by Hume's Principle in providing neo-Fregean foundations for elementary arithmetic. The corresponding formal result is that Hume's Principle plus second-order logic suffices for the construction of an ω sequence. That is certainly of mathematical interest. But what gives Frege's Theorem its distinctive and additional *philosophical* interest is that Hume's Principle also purports to give an account of *what cardinal numbers are*. The philosophical payload turns not on the mathematical reduction as such but on the specific content of the abstraction by which the reduction is effected.

To enlarge: Hume's Principle effectively incorporates a variety of philosophical claims about the nature of number for which Frege prepares the ground philosophically in the sections of *Grundlagen* preceding its first appearance—for example, the claims (1) that number is a second-level property, a property of concepts, and it is concepts that are the things that *have* numbers, which is incorporated by the feature that the cardinality operator is introduced as taking concepts for its arguments; and (2) that the numbers themselves are objects, which is incorporated by the feature that terms formed using the cardinality operator are singular terms. And in addition, of course, Hume's Principle purports to explain (3) what sort of things numbers are. It does so by framing an account of their criterion of identity in terms of when the things that have them have the same one: numbers, according to Hume's Principle, are the sort of things that concepts share when they are one-to-one correspondent.

Now one could, if one wanted, read a corresponding set of claims about real number off the Cut Abstraction principle featured in the Dedekindian Way. One would then conclude, correspondingly, that real numbers are objects, that the things which have real numbers are properties of rationals, and that real numbers are the sorts of things that properties of rationals share just when their instances have the same rational upper bounds. One could draw these conclusions. But, apart from the first, they are unmotivated-seeming conclusions to draw. There is no philosophical case that real number is a property of properties of rationals which stands comparison with Frege's case that cardinal number is a property of

(sortal) concepts. On the contrary, the intuitive case is that real number belongs to things like *lengths*, *masses*, *temperatures*, *angles*, and *periods of time*. We could conclude that the Dedekindian Way incorporates poor answers to questions whose analogues about the natural numbers Hume's Principle answers relatively well. But a better conclusion is that the Dedekindian Way was not designed to take *those* kinds of questions on.

In order to understand what is at stake here, we need to appreciate that Hume's Principle accomplishes two quite separate foundational tasks. There is, a priori, no particular reason why a principle intended to incorporate an account of the nature of a particular kind of mathematical entity should also provide a sufficient axiomatic basis for the standard mathematical theory of that kind of entity. It's one thing to characterize what kind of entity we are concerned with, and another thing to show that and why there are all the entities of that kind that we standardly take there to be, and that they compose a structure of the kind we intuitively understand them to do. The two projects may be expected to interact, of course. But they are distinct. It is a peculiar feature of the standard neo-Fregean foundations for elementary arithmetic that the one core principle, Hume's Principle, discharges *both* roles. This is not a feature which we should expect to be replicated in general when it comes to providing neo-Fregean foundations for other classical mathematical theories. What the reflections of a moment ago suggest is that the Dedekindian Way, for its part, is best conceived as addressing only the second project.

It is the distinction between these two projects—the metaphysical project of explaining the nature of the objects in a given field of mathematical inquiry and the epistemological project of providing a foundation for the standard mathematical theory of those objects—that, so far as one can judge from the incomplete discussion in *Grundgesetze*, seems to have been a principal determinant of the approach taken by Frege himself. Real numbers, as remarked, are things possessed by lengths, masses, weights, velocities, and such—things which allow of some kind of magnitude or *quantity*. Quantities are not themselves the reals, but the things which the reals *measure*. As Frege says:

… the same relation that holds between lines also holds between periods of time, masses, intensities of light, etc. The real number thereby comes off these specific kinds of quantities and somehow floats above them. (*Grundgesetze* §185)

So if we want to formulate an abstraction principle incorporating an answer to the metaphysical question "What kind of things are the reals?" after the fashion in which Hume's Principle incorporates an answer to the metaphysical question "What kind of things are the cardinal numbers?" then quantities will feature, not as the domain of reference of the new singular terms which that abstraction will

introduce, but rather as the abstractive domain—that is, as the terms of the abstractive relation on the right-hand side.

However, it's clear that individual quantities don't have their real numbers after the fashion in which a particular concept, say *Julio-Claudian Emperor of Rome*, has its cardinal number. We are familiar with different systems of measurement, like the imperial and metric systems for lengths, volumes, and weights, or the Fahrenheit and Celsius systems for temperature, but there is no conceptual space for correspondingly different systems of counting. Of course, there can be different systems of counting *notation*: we can count in a decimal or binary system, for instance, or in Roman or Arabic numerals. But if they are used correctly, they won't differ in the cardinal number they deliver to any specified concept, but only in the way they name that number. By contrast, the imperial and metric systems do precisely differ in the real numbers they assign to the length of a specified object. One inch is 2.54 centimeters. The real number properly assigned to a length depends on a previously fixed unit of comparison. Thus real numbers are *relations* of quantities, just as Frege says.

These reflections seem to enforce a view about what a principle would have broadly to be like whose metaphysical accomplishment for the real numbers matches that of Hume's Principle for the cardinals. Where Hume's Principle introduces a monadic operator on concepts, our abstraction for real numbers will feature a dyadic operator taking as its arguments a pair of terms standing for quantities of the same type; more specifically, it will be a *first-order* abstraction:

- *Real Abstraction* $\mathrm{R}\langle a,\ b\rangle = \mathrm{R}\langle c,\ d\rangle \leftrightarrow \mathrm{E}(\langle a,\ b\rangle,\ \langle c,\ d\rangle)$,

where *a* and *b* are quantities of the same type, *c* and *d* are quantities of the same type (but not necessarily of the type of *a* and *b*), and E is an equivalence relation on pairs of quantities whose holding ensures that *a* is proportionately to *b* as *c* is to *d*. In effect, the analogy is between the abstraction of cardinal numbers from one-one correspondence on concepts, and abstraction of real numbers from equiproportionality on pairs of suitable quantities.

A neo-Fregean who wishes to pursue this antistructuralist, more purely Fregean, conception of a foundation for analysis must therefore engage with each of the following three tasks.

First, a philosophical account is owing of what in the first place a *quantity* is—what the ingredient terms of the abstractive relation on the right-hand side of the Real Abstraction principle are.

Second—if the aspiration is to give a logicist treatment in the extended sense in which Hume's Principle provides a logicist treatment of number theory—it

must be shown that, parallel to the definability of one-one correspondence using just the resources of second-order logic, both the notion of *quantity* and the relevant equivalence relation, E, allow of (ancestral) [44] characterization in (second-order) logical terms. [45]

Third, a result needs to be established analogous to Frege's Theorem: specifically, it needs to be shown that there are sufficiently many appropriately independent truths of the type depicted by the right-hand side of Real Abstraction to ground the existence of a full continuum of real numbers. And while, as stressed, Hume's Principle itself suffices for the corresponding derivation for the natural numbers, here it is clear that additional input—the construction of a complete domain of quantities—is going to be required to augment the Real Abstraction principle. [46]

We have no space here to go further into the philosophical issues at stake between the Dedekindian Way and the more purely Fregean approach. A key question is the cogency, in the context of foundations for analysis, of what we term *Frege's Constraint*: that a philosophically satisfactory foundation for a mathematical theory must somehow intimately build in its possibilities of application. This may be taken to require that an abstraction to the special objects of a mathematical theory is satisfactory only if the abstractive domain on its right-hand side comprises the kinds of things that the mathematical objects in question are *of*—the kinds of things of which they provide a mathematical measure. A structuralist may be expected to reject Frege's constraint, or anyway this understanding of it. The matter is open. [47]

7. Neo-Fregean Set Theory

Much further work is needed here. We give but the briefest sketch of some salient approaches and their problems.

An abstraction principle which can plausibly be seen as implicitly defining the concept of *set* will do so by fixing the identity conditions for its instances, and will—on pain of changing the subject—take the identity of sets to consist in their having the same members. So if we assume—what does not seem seriously disputable—that any plausible candidate will be a higher-order abstraction, then the equivalence relation involved had better be coextensiveness of concepts. What we are looking for is, accordingly, a consistency-preserving restriction of Basic Law V, where the restriction is given by imposing a constraint on which concepts are to have properly extensionally behaved sets corresponding to them. [48]

Let's schematically represent the sought-after constraint using a second-level predicate "Good." Then two natural ways to restrict Basic Law V are

- (A) $\forall F \forall G[(\text{Good}(F) \wedge \text{Good}(G)) \rightarrow (\{x \mid Fx\} = \{x \mid Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx))]$

and

- (B) $\forall F \forall G[(\{x \mid Fx\} = \{x \mid Gx\} \leftrightarrow (\text{Good}(F) \vee \text{Good}(G)) \rightarrow \forall x(Fx \leftrightarrow Gx))]$.

The main difference between these is that (A) is a conditionalized abstraction principle, whereas (B) is unconditional, with the restriction built into the relation required to hold between *F* and *G* for them to yield the same set—fairly obviously, the resulting relation is an equivalence relation. [49] Consequently, (B) yields a "set" for every *F*, regardless of whether it is Good or not, whereas (A) yields a set $\{x|Fx\}$ only if we have the independent input that *F* is Good. If neither *F* nor *G* is Good, the right-hand side of (B) holds vacuously, so we get that $\{x|Fx\} = \{x|Gx\}$—regardless of whether *F* and *G* are coextensive. That is, we get the same "set" from all Bad concepts. We get real sets via (B)—that is, objects whose identity is determined by their membership—only from Good concepts.

So what might it be for a concept to be Good? Various suggestions have been canvassed. One general approach, first proposed by Boolos, [50] picks up on the well-entrenched "limitation of size" idea, that the set-theoretic paradoxes stem from treating as sets "collections" which are in some sense "too big"—the collection of all sets, of all sets that are not members of themselves, of all ordinals and so on. Following Boolos, define a concept *F* to be *Small* = Good if there is a one-one function taking the *F*s into the concept *self-identical* but no one function taking the self-identicals into the *F*s. The resultant restricted set abstraction in the style (B) is what Boolos called New V (and Wright called VE): [51]

- $New\ \ V\ \forall F \forall G[\{x|\ Fx\} = \{x|\ Gx\} \leftrightarrow (\mathrm{Small}(F) \vee \mathrm{Small}(G)) \rightarrow \forall x(Fx \leftrightarrow Gx))].$

Boolos showed that a significant amount of set theory can be obtained in the system consisting of New V and second-order logic. [52] But there are two serious problems with it. The first is that we don't get *enough* set theory. As Boolos showed, neither an axiom of infinity (that there is an infinite, properly behaved *set*) nor the power set axiom can be obtained as theorems on this basis—so the theory affords not even a glimpse of Cantor's Paradise.

A further, quite different, problem relates to the constraints needed to differentiate between acceptable abstraction principles and unacceptable ones. Earlier we noted the need for a constraint of a certain kind of *conservativeness*: acceptable abstraction principles should do no more than fix the truth-*conditions* of statements involving the entities whose concept they introduce. They should have nothing to say about the truth-*values* of statements whose ontology is restricted to entities of other kinds. The Nuisances abstraction, for instance, was faulted on precisely these grounds. Unfortunately, New V appears to be in difficulty for the same reason. The basic point is simple enough. [53] If the concept *ordinal* is Small, New V yields a properly behaved set of all the ordinals and will therefore generate the Burali-Forti contradiction. Hence *ordinal* must be Big. But in that case it is exactly as big as the universe (i.e., there is a one-one correspondence between *ordinal* and a(ny) universal concept, say self-identity. And

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this, together with the fact that the ordinals are well-ordered by membership, entails global well-ordering—the existence of a well-ordering of the universe, and hence of any subuniverse consisting just of the ontology of a prior theory: a result which may well be independent of a suitably chosen such theory. So New V must be reckoned nonconservative.

This problem may be remediable by a suitable redefinition of Smallness. Reinterpret Goodness as *Double Smallness*, where a concept is doubly small if and only if it is (strictly) smaller than some concept which is itself (strictly) smaller than some concept; that is:

- $\text{Small}^2(F) \leftrightarrow \exists\, G \exists\, H(F < G < H)$.

This blocks the reasoning which shows that New V, as originally understood, implies global well-ordering. We can still show, of course, that *ordinal* cannot be Good—that is, now, Small^2—since if it were, we would have the Burali-Forti paradox, just as before. So we have to agree that *ordinal* is Bad. But that just means that it is not Small^2, and from this we cannot infer that it is bijectable onto any universal concept (or, indeed, onto any concept). [54]

The other, and more serious, of the two problems New V faces, as we noted, was its weakness: its inability to deliver either an axiom of infinity or a power set axiom as theorems. The same will go, of course, for New V reinterpreted with Good as Small^2 and Small^2 V. However, this need not be a crippling drawback from the neo-Fregean's point of view, if he can justify supplementing New V, or Small^2 V, with other principles—most especially, other abstraction principles—which compensate. On this more catholic approach, we separate two distinct roles one might ask a set-abstraction principle to discharge: *fixing the concept* of set, on the one hand, and, on the other, serving as a *comprehension principle*. The claim would then be that New V's, or Small^2 V's, shortcomings as a comprehension principle need not debar it from successfully discharging a concept-fixing role—of

serving as a means of introducing the concept, while leaving its extension to be filled out, largely or even entirely, [55] by other principles. An obvious first port of call for this approach would be to consider a system consisting of second-order logic augmented by, say, Small^2 V, Hume's Principle, and an appropriate range of instances of a schematic form of the Cut Abstraction principle. Notice that it would be perfectly legitimate, for the purposes of this project, to restrict the domain of the $<$ relation, featured in the definition of Small^2, to concepts instantiated exclusively by the objects given by these privileged prior abstractions. [56]

A quite different direction, but still in keeping with the overall approach, would be to interpret Goodness not in terms of size but in terms of *definiteness*. Here definite concepts contrast with those which Michael Dummett has termed "indefinitely extensible." An indefinitely extensible concept is one, roughly, for which any attempt at an exact circumscription of the extension immediately gives rise to further intuitively acceptable instances falling outside that circumscription. One classic line of diagnosis of the set-theoretic paradoxes, originating in effect with Russell, has it that they arise through reasoning about indefinitely extensible concepts such as *ordinal number*, *cardinal number*, and *set* itself as if they were definite, and in particular by unguardedly assuming that such concepts give rise to collections of entities apt to constitute *sets*. If it were possible to give sufficient of an exact characterization of indefinite extensibility to make that thought good, there would be both a sense of having accomplished a solution to the paradoxes and a clear motive for an appropriate version of either schema (A) or schema (B) which restricted the concepts determining sets to those which are definite. And if the sought-for characterization could somehow be given using only resources available in higher-order logic, we might have the basis for a strong and well-motivated neo-Fregean set theory.

It must be confessed that the hope at present seems somewhat utopian. No sufficiently clear account of the notion of indefinite extensibility, still less one deploying only logical resources, has yet been achieved. And even if one can be, it is by no means clear in advance that it will not be beset by problems of weakness analogous to those of the various Smallness abstractions. The crucial question, as far as overcoming the problem with infinity is concerned, is whether there is any

acceptable account of indefinite extensibility which, while embracing the usual suspects—*ordinal*, *cardinal*, and *set*—ranks *natural number* as definite. Work so far has not found one. And even if that trick can be turned, it will be necessary that the characterization of natural number as definite on that basis can proceed on the basis of considerations formalizable in higher-order logic. Otherwise there will be no payoff in terms of the proof-theoretic strength of the abstraction principle in question.

8. Neo-Fregean Logic

The classic logicist thesis about a particular mathematical theory is that its fundamental laws are obtainable on the basis just of definitions and logic. It is still, at the time of writing, a justifiable complaint that while much attention has been paid by neo-Fregeans, and their critics, to the first component in the recipe—issues to do with abstractions in general and Hume's Principle in particular—comparatively little has been given to the second component: the demands, technical and philosophical, to be made on the logical system which is to provide the medium for the proofs the neo-Fregean needs. That system will be either, to all intents and purposes, the higher-order logic pioneered in Frege's *Begriffsschrift* or some substantial fragment of it. Even if it is granted that all the outstanding questions concerning abstractions may be settled in the neo-Fregean's favor, the interest of the neo-Fregean reconstructions of classical mathematical theories will very substantially depend upon their being possible to return logicist-friendly answers to a number of fundamental questions about logical systems of this kind.

The significance which a successful logicist treatment of a particular mathematical theory was traditionally believed to carry was based on the assumption that logic is interestingly set apart from, and somehow more fundamental than, other formal a priori disciplines—that it is marked off by, for instance, its generality, its implication in anything recognizable as rational thought, and by the special character of its distinctive vocabulary—the connectives and other logical constants. It is only in the setting of an acceptance that logic is, in some such way, metaphysically and epistemologically privileged that a reduction of mathematical theories to logical ones—more accurately, to logical theories and legitimate (implicit) definitions—can be philosophically any more noteworthy than a reduction of any mathematical theory to any other. The great issue raised is therefore whether the philosophy of logic can indeed provide a clear account answering to our sense of the fundamentality of logic and its separation from other a priori disciplines and, more specifically, whether—in the context of such an account—higher-order logic will indeed be classified as properly so conceived.

A key issue here concerns the *ontology* of higher-order logic. When Quine famously quipped that he regarded higher-order logic as "set theory in sheep's clothing," his point was not that, in his view, set theory might have a case to count as logic; it was, rather, that higher-order logic had better be regarded as mathematics. However, someone who was persuaded that Quine was wrong about that—that, qua logic, higher-order logic loses nothing to first-order logic—might still be concerned about the ontology apparently demanded by higher-order quantification. Such a thinker might thus be persuaded that there was a *logicist* insight to be recovered by neo-Fregeanism but still thoroughly skeptical whether anything to render the ontology of traditional mathematical theories more palatable can be achieved by the neo-Fregean's abstractionist story, the invocation of the Context Principle, and the rest, when the logical component of the reducing theory—higher-order logic plus abstraction principles—incorporated such indigestible ontological presuppositions of its own.

A frequent type of response would be to accept the problem at face value—to accept that any ontology which was allowed to be distinctively involved in higher-order quantification would indeed be problematical—and try to maintain that in contrast to first-order quantification, higher-order quantification actually incorporates no special ontological commitments: that it is best construed substitutionally, for instance. Or else second-order quantification might be understood in the manner, explored by George Boolos, whereby the only entities involved are those already quantified over at first order (viz., objects). The meaning of higher-order quantifiers is determined by postulating an analogy between their relationship to plural denoting expressions and the relationship between first-order quantification and ordinary singular terms. [57] Each of those proposals, however, would have important limitations from the neo-Fregean perspective. A substitutional interpretation of higher-order quantification looks obviously insufficient to sustain the demands placed on higher-order logic by the kind of neo-Fregean treatment of analysis canvassed above, for instance, while Boolos's construal extends no further than existential quantification into the places marked by monadic predicates. [58]

Ingenuity may come up with further possibilities. For instance, it might be possible to reduce relational quantification, of arbitrary degree, to monadic quantification in the context of a first-order ontology enriched by appropriate *n*-tuples of objects, yielded in turn by appropriate first-order abstraction principles. [59]

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But without prejudging the prospects for such reductionist approaches, we ourselves are inclined to favor a different direction. The thought is intuitively compelling that quantificational statements cannot import a *type* of ontological commitment that is not already present in their instances. Thus, if the quantifiers of higher-order logic were indeed best interpreted as calling for an ontology of sets, or setlike entities of some sort, the same should be said about the simple predications which instantiate them. Alternatively, it might be argued that since simple predication, unlike singular reference, incorporates no special ontology, higher-order quantification doesn't either. [60] That thought might be taken to demand a substitutional interpretation of higher-order quantification. But there are at least two other possibilities to explore. First, suppose we can lucidly conceive of thoughts—propositions—as literally internally structured entities. [61] Then we can grasp the idea of one thought resulting from another merely by variation on some internal structural theme they both exhibit, and perhaps also the idea of *all* thoughts which so result (or at least, all which satisfy some other specific restriction). And in that case there has to be a way of understanding quantification which liberates it from the idea of *range*—the association with a domain of entities of some kind or other which is the (Quinean) root of the ontological problems with higher-order logic. Simply, let

$(\forall F)Q(F)$

be the weakest thought whose truth suffices for that of any thought accessible by playing the prescribed kind of variation on the place marked by the dots in

Q …

Likewise, (∃*F*)Q(*F*) will be the strongest thought for whose truth the truth of any such thought suffices.

This construal has a kind of substitutional flavor, but it doesn't have the usual limitation of substitutional quantification of being confined to a particular linguistic repertoire. It can literally be of unrestricted generality, or as unrestricted a generality as makes thinkable sense. Whether higher-order quantification implicates an ontology now depends on whether or not the kind of "thought-component" for which the dots mark a place does. If that component involves direction upon an object—the thought-analogue of singular reference—then of course it will. So first-order quantification does. But it is only if one has argued independently that the thought-components that correspond to predication are entity-involving that higher-order quantification will be entity-involving.

The other possibility is more Fregean and more squarely in keeping with the neo-Fregean approach to abstract ontology in general. It is to apply the Context Principle, conceived as a principle concerning *Bedeutung*, to incomplete expressions as well as to complete ones. This approach would hold that, just as the occurrence of a singular term in a true (extensional) statement demands an object as its referent, so the occurrence of an *n*-place predicate in such a statement demands a concept under which the object(s) referred to therein are thereby brought. And just as, in the case of reference to abstract objects, any sense of epistemic impasse is to be offset by explaining what is to refer to and know about such objects in terms of competence in the discourse in which their names occur, so the epistemology of the entities answering to predication would be accounted for in terms of competence in the use of the predicates associated with them. In brief, the proposal would be that higher-order logic should be accepted at face value, as quantifying over entities which constitute the semantic values of incomplete expressions, and that worries about the character of such entities, and about their epistemic acceptability, are to be addressed in exactly the same kind of way, mutatis mutandis, as the way in which neo-Fregeanism approaches the issues to do with abstract singular reference. Of course, a number of issues immediately loom large, not least the paradox of "the Concept *Horse*," the nature of the semantic relationship between incomplete expressions and the entities over which higher-order logic quantifies, and how (finely) these entities are individuated. But perhaps enough has been said to indicate why someone who is content that the neo-Fregean approach to abstract singular reference is broadly along the right lines might be hopeful that no specially opaque or intractable issue might be posed by higher-order quantification as such.

Finally, one more technical matter arising in this context concerns the varying demands placed on the underlying logic by different phases of the neo-Fregean reconstruction of mathematics. J. L. Bell (1999) has shown that the higher-order logic necessary for the demonstration of Frege's Theorem on the basis of Hume's Principle is in fact quite weak. [62] By contrast, it's obvious enough that any abstraction which is to generate a classically uncountable infinity of objects—like that canvassed for the real numbers—must draw on a classically uncountable higher-order domain. So the intriguing and difficult issue therefore arises whether it is possible to justify a conception of the higher-order ontology which can sustain this demand; and whether, in particular, it is possible to do so on the basis
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of the approaches to higher-order quantification canvassed a moment ago, which inevitably effect a very close tie between (*n*-adic) *concept* and *(item of a) thinkable predication*. The possibility must be taken seriously, it seems to us, that the most tractable philosophies of higher-order logic will have the side effect that a neo-Fregean treatment of analysis will be constrained to be *constructivist*, and that the classical continuum will be out of reach. (Whether that would be cause for regret is, of course, a further question.)

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7. Logicism Reconsidered

Agustín Rayo

This chapter is divided into four sections.* The first two identify different logicist theses, and show that their truth-values can be established given minimal assumptions. Section 3 sets forth a notion of "content-recarving" as a possible constraint on logicist theses. Section 4—which is largely independent from the rest of the paper—is a discussion of "neologicism."

1. Logicism

1.1. What Is Logicism?

Briefly, logicism is the view that mathematics is a part of logic. But this formulation is imprecise because it fails to distinguish among the following three claims:

1. *Language-Logicism*
   The language of mathematics consists of purely logical expressions.

2. *Consequence-Logicism*
   There is a consistent, recursive set of axioms of which every mathematical truth is a purely logical consequence.

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3. *Truth-Logicism*
   Mathematical truths are true as a matter of pure logic.