12. See *Articles on Aristotle 4. Psychology and Aesthetics*, ed. Jonathan Barnes, Malcolm Schofield and Richard Sorabji (Duckworth, 1979), pp. 65–75. One response to Ackrill's criticism of hylomorphism can be found in Jennifer Whiting's 'Living Bodies' in *Essays on Aristotle's De Anima.*
13. *Essays on Aristotle's De Anima*, pp. 343–358.
14. See 'Minds and Machines' in *Dimensions of Mind,* ed. Sidney Hook (Collier, 1960).
15. This account is lifted from Ned Block's useful discussion of functionalism in *Readings in Philosophy of Psychology* Vol. 1 (Harvard University Press, 1980). Block cites Aristotle as the first functionalist philosopher of mind.
16. Hartman's position is outlined in *Substance, Body and Psuche* and Martha Nussbaum's most recent article is 'Changing Aristotle's Mind', co-written with Hilary Putnam in *Essays on Aristotle's De Anima.*
17. Burnyeat's paper circulated, unpublished, for almost a decade and finally appeared in print in 1992 in *Essays on Aristotle's de Anima* edited by Martha C. Nussbaum and Amelie Oksenberg Rorty.
18. See 'Changing Aristotle's Mind' by Martha C. Nussbaum and Hilary Putnam, and 'Hylomorphism and Functionalism' by S. Marc Cohen in *Essays in Aristotle's De Anima.*
19. For an interpretation of Aristotle's theory of soul that is not functionalist but also thinks that it does not belong in the trash heap of history see 'Explaining Various Forms of Living' by Alan Code and Julius Moravcsik in *Articles on Aristotle's De Anima.*
20. Robinson's paper is in *Oxford Studies in Ancient Philosophy,* vol. i, 1983; a reply to it by Martha Nussbaum is in the next volume.
21. In *Essays on Aristotle's De Anima.*
22. In 'Aristotle's Perceptual Realism' *Ancient Minds,* vol. xxxi, Supplement to *The Southern Journal of Philosophy.*
23. For the materialist view see 'Aristotle on Sense-Perception' by Thomas Slakey in *Aristotle's De Anima in Focus*, ed. Michael Durrant (Routledge, 1992). Sorabji's alternative account is 'Intentionality and Physiological Processes: Aristotle's Theory of Sense-Perception', in *Essays on Aristotle's De Anima.*
24. See 'A New Look at Aristotle's Theory of Perception' in *Aristotle's De Anima in Focus.*
25. Nussbaum's account of *phantasia* is in *Aristotle's De Motu Animalium* (Princeton University Press, 1978) and Schofield's paper 'Aristotle on the Imagination' has been reprinted in *Essays on Aristotle's De Anima.*
26. In *Essays on Aristotle's De Anima.*

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CRITICAL NOTICE

FREGE: PHILOSOPHY OF MATHEMATICS

This book (*Frege: Philosophy of Mathematics*, by Michael Dummett. Duckworth, 1991. xii + 321 pp. £35.00) is the sequel to Dummett's *Frege: Philosophy of Language* (*FPL*), first published twenty-one years ago. But anyone expecting some seven hundred or so pages in which the central ideas of Frege's writings about mathematics, refurbished and extended, would form the basis for a treatise on all, or almost all issues of contemporary concern in the philosophy of mathematics, will be disappointed. Less than half the length of *FPL*, *Frege: Philosophy of Mathematics* (*FPM*) contrasts with its predecessor in its virtual confinement to Frege's actual agenda in the philosophy of mathematics—the problems of the status of the truths of number theory and analysis, and of our knowledge of them. One effect of this is that much of the book is given to the kind of critical analytical commentary on Frege's texts which was conspicuously absent from *Frege: Philosophy of Language*. It is no criticism of the former book to welcome this. In particular, the secondary literature on Frege has hitherto lacked anything to compare with the detailed account of the argumentation of the Introduction and first three sections (up to § 55) of *Grundlagen* which Dummett offers over the first third of his book.

It ought to be no surprise that the two books should so contrast in scale, organisation and philosophical style. For in the Introduction of *FPL* Dummett wrote, recall, that

> [Frege's] philosophy of mathematics is [now] a starting point only in a historical sense. It is not true that Frege's formulation of the problems of the philosophy of mathematics any longer strikes us as the most fruitful way to pose the questions that arise on the subject: on the contrary, certain questions which subsequent writers have caused to appear as constituting the central issues passed Frege by as if quite unproblematic (*FPL*, p. xxxiv)

And while

> Frege's work in philosophical logic remains fresh and relevant to the problems as we now seen them. (*Ibid.*)

The verdict must be that

> Frege, as a philosopher of mathematics, is indisputably archaic in a sense in which Hilbert and Brouwer are not. (*FPL*, p. xxxviii)

What *is* surprising is that *FPM* is much more than the project of historical criticism to which these remarks would seem to restrict it. If, in 1973, Dummett held not merely that Frege's mathematical philosophy was full of error, but that it was addressed to questions of questionable centrality, he does not, in 1993—or not consistently—claim either. True, there are passages, like that which speaks of "the central flaw in [Frege's] entire philosophy of arithmetic" (*FPM* p. 232. All references are to *FPM* unless otherwise stated.) and that which alleges that "[Frege's] view of the status of the numbers, ontological and epistemological, prove to be catastrophically wrong" (p. 292),[1] which seemingly write off the core of Frege's philosophy of arithmetic. But Dummett does not intend to dismiss the logicist thesis, nor to deny the Context Principle a vital role in validating the use of abstract mathematical singular terms. While there is, I think, considerable unclarity about what precisely Dummett believes can be salvaged from the wreck, Frege is now credited with a driving insight into the nature of mathematics, encapsulated in the logicist thesis, which, in Dummett's view, has still to be improved, upon by the work of later philosophers. He writes

> For [Frege], the whole point of mathematics lay in its applications. A mathematical theorem, on his view, encapsulates an entire deductive subroutine—perhaps a very complex one—which, once discovered, does not need to be gone through again explicitly on future occasions; but it expresses it, not as a principle of inference, but as a proposition to which we have given sense by fixing its truth-conditions, which may therefore be considered on its own account, without an eye on its possible applications. On this view,...that part of mathematics which is independent of intuition simply comprises all the complex deductive reasoning of which we are capable, purged of all that would restrict its application to particular realms of reality . . . Geometry apart, mathematics therefore simply *is* logic: no distinction in principle can be drawn . . . It is difficult to maintain that any more convincing account of the general nature of mathematics has ever been given. (pp. 308–9)

It does not seem likely that the 'archaism' passages of the 1973 Introduction were penned with this assessment in mind. Rather the intervening years would seem to have seen a substantial change of mind about the proper assessment of Frege's writings about mathematics.

II

In briefest overview, *FPM* is structured as follows. Chapter 1 briefly reviews Frege's philosophical development and lays down the ground-rule that the informal philosophy underlying the system of *Grundgesetze* must, to the greatest consistent extent, be taken to be that of *Grundlagen*.[2] Chapters 2–5 are concerned with the overall nature of Frege's project, focussing on Frege's methodology as avowed in the Introduction to *Grundlagen*, his anti-psychologism and his conception of analyticity.[3] Chapter 4 in particular highlights Frege's ideas concerning the question how deductive inference, and analytic propositions, can be news. Frege's answer, commended by Dummett, is that recognising valid inference, and analyticity in general, may demand finding a syntactic form in a sentence not involved in grasping its meaning.[4] Chapter 5 instructively compares Frege's project and approach with that of Dedekind's *Was sind und was sollen die Zahlen?*.

Chapters 6–9 offer a rigorous critical commentary on parts II and III of *Grundlagen*—the sections in which Frege so ruthlessly dissects the then available philosophical ideas about number, working towards his famous comparison of judgements of number with judgements of existence (*Grundlagen* § 53). Chapter 9, however, on the first seven sections of part IV of *Grundlagen*, is severely critical of the grounds Frege there offers for, in effect, rejecting the conception of numbers as second level concepts (quantifiers) in favour of the thesis that they are to be taken as objects.[5]

Chapters 10–14 are devoted to *Grundlagen* §§ 62–69—"the most brilliant and philosophically fruitful [sections] in the book" (p. 111). Here Frege sets out the key moves in his philosophy of arithmetic:

> The application of the Context Principle to license the thought that an epistemology of mathematical objects may be accomplished by an epistemology of mathematical statements;
>
> The selection of the central case of identity statements featuring such terms;
>
> The canvassing of the strategy of contextual introduction of such statements by reference to antecedently understood equivalence relations on antecedently understood domains;[6]
>
> The introduction of the special case whereby statements of numerical identity are fixed as equivalent to statements concerning relations of one-one correspondence among concepts;[7]
>
> The sustaining of the Caesar problem as a fatal objection to the contextual approach;
>
> And, finally, Frege's famous response to the Caesar problem, viz. to identify the cardinal numbers with equivalence classes of concepts under the relation "may be put into one-one correspondence with".

Dummett's discussion of these classic moves is brisk but rich in detail. It includes (in Chapter 12) an illuminating review of Husserl's objections to Frege's procedures, most especially concerning issues of priority among the ingredients in his various definitions, in the former's *Philosophie der Arithmetik*.

Chapters 15–19 are devoted to Frege's ontology: to the interpretation of the Context Principle, in *Grundlagen* and in *Grundgesetze*, and to whether it can indeed be deployed to explain how numbers "may be given to us". These sixty or so pages encompass the core of Dummett's objections to Frege's procedure. Overshadowing the discussion is the question: why did Frege's project culminate in inconsistency—an avoidable blunder, or the symptom of an irremediable flaw in the entire project? Dummett's view is the latter. Frege fell into contradiction because he did not scruple to let his constructions rely upon impredicative quantification, whereby we supposedly specify particular objects, or concepts, by quantifying over an objectual, or conceptual domain which is taken to include those very items. For Dummett, such a procedure is open to the charge of circularity even in cases where no actual contradiction results. The error is not in the general methodology of the Context Principle, but in Frege's implementation of it:

> What must ... be disputed is Frege's ... idea of what is sufficient for determining the truth-conditions of sentences containing terms of a newly introduced kind. Impredicative specification of the conditions for the truth of identity statements involving one or two such terms is *not* sufficient ... it fails to fix truth-conditions for all sentences containing the new terms, when these terms are formed by attaching an operator to a predicate or

> functional expression and it fails to do so because of a lack of an independent specification of the domain, which it attempts, but fails, to circumscribe simultaneously with its determination of the truth-conditions of sentences containing the new terms. (p. 236)

That terse formulation goes to the heart of Dummett's objection both to Frege's procedure and to the approach which I canvassed on Frege's behalf in my *Frege's Conception*. I'll return to it below.

Chapters 19–22 are concerned with part III of *Grundgesetze* and Frege's theory of real numbers, and include detailed discussion of Frege's critique of formalism and of Cantor. Dummett rightly places special emphasis on Frege's insistence that a satisfactory account of statements concerning real numbers must build in their application to physical magnitudes as an intrinsic part of their content. This is the analogue of the requirement that a satisfactory account of statements of cardinal number should explain the applicability of such statements to objects as diverse as symphonies, years, apples and the numbers themselves.

Finally Chapters 23 and 24 offer assessment of the overall accomplishment of Frege's writings about mathematics, and propose a view of his general legacy. The balance-sheet, very roughly, turns out as follows:

Positive

The successful polemic against formalism;

The rightful emphasis on the applications of pure mathematics as something intrinsic to its content, and the significant strides taken towards an account of the content of statements of number theory and analysis which explains their applicability;

The explanation of how analytic truths and deductive inference can extend our knowledge;

The insight, contained in the Context Principle, that reference to abstract objects need not per se be problematical;

The overarching logicist insight into the general character of pure mathematics.

Negative

The failure to see that there is an issue about the validity of classical logic applied to mathematical theories (something for which, however, in Dummett's view, Frege "can hardly be blamed" (p. 321));

The "big error" (*Ibid.*), the fundamental flaw, viz. the misdeployment of the Context Principle in justification of principles, including the notorious Basic Law V, which embodied vicious circularity and led to the collapse of his entire system.

Whether that adds up to something which justifies the concluding accolade—"the greatest philosopher of mathematics yet to have written" (*Ibid.*)—must depend on whether Dummett has pointed to a way of avoiding the "big error" while retaining the logicist insight and a broadly Fregean conception of the epistemology of mathematical objects. I shall come back to this.

III

In §§ 72–83 of *Grundlagen*, Frege outlines definitions and informal reasoning which strongly suggest that proofs of the fundamental laws of arithmetic—the axioms for successor and the principle of mathematical induction—are available in second order logic with identity with $N^=$ as the sole additional postulate, and without any reliance on extensions of concepts or the ill-fated Basic Law V. Charles Parsons affirmed that this is so;[8] and in the concluding section of *Frege's Conception*, I showed in some detail how it is so.[9]

That $N^=$ entails the usual axioms of arithmetic has recently come to be known as *Frege's Theorem*.[10] The appraisal of Frege's philosophy of arithmetic must be, in large measure, the appraisal of the philosophical significance of Frege's Theorem. What makes the theorem particularly interesting are the similarities, and one crucial difference, between $N^=$ and Basic Law V. Both principles abstract from a second order equivalence relation—an equivalence relation on concepts—to the conditions of the identity of a new kind of object; and each second order equivalence relation may be defined using just the resources of second order logic with identity. The difference is that $N^=$ is known to be consistent.[11] May it therefore be accorded the status which Frege believed was enjoyed by Basic Law V? If so, then whatever *Grundgesetze* would have shown about arithmetic and analysis, had Basic Law V not been inconsistent, *is* shown to be true of arithmetic at least by the establishment of Frege's Theorem.

In *Frege's Conception of Numbers as Objects* I argued that Frege's Theorem is indeed at the service of a modest platonist logicism: that $N^=$ should be viewed not, indeed, as a truth of logic, nor as a definition in a strict sense, but rather as the core of an *explanation* of the notion of cardinal number, which may rank as an analytic truth in the same way, and for the same reason, that any explanatory definition—any definition used to explain a new concept—does. If this is correct, then Frege's Theorem shows that the basic laws of arithmetic are analytic. Moreover, since the concept formation embodied in $N^=$ proceeds on the basis solely of the conceptual repertoire of second order logic with identity, there is a case for a version of logicism, viz, that Frege's Theorem shows not that the truths of arithmetic are, in effect, truths of logic, but that they are consequences of an analytically explanatory principle which may be laid down—like any consistent explanation—without further epistemological presupposition, and the grasp of which requires only logical notions. In that sense, there is a route from logic to arithmetic which does explain how the natural numbers "may be given to us as objects"[12] and does carry something close to the epistemological significance which Frege's logicist thesis, as more usually understood, was intended to bear.[13]

Dummett is utterly out of sympathy with this particular version of the neo-logicist proposal. Criticisms of it are rife throughout the latter two-thirds of *FPM*, and may be grouped under four broad heads:

(i) In the chapter on 'Abstract Objects' in *FPL*, and others of his previous writings,[14] Dummett contended that while a use of the Context Principle to justify the ascription of reference to abstract singular terms may correctly counter the nominalist stance, what is vindicated thereby is only a 'thin' (contrast: *realistic*) conception of reference for such terms. The coherence of this claim was questioned in some detail in *Frege's Conception*.[15] Dummett now repeats the claim—or something very close to it[16]—insisting that the required distinction between two conceptions of reference is valid, and that there is no justification for any but a "non-realistic" notion of reference for contextually introduced singular terms.[17]

(ii) The core of the Fregean view is that a referential construal of a range of abstract singular terms may be explained and justified by abstractive stipulation—by laying down equivalences like $N^=$, or the corresponding principle for the identity of directions,

For all lines *a*, *b*: the direction of *a* = the direction of *b* $\leftrightarrow$ *a* and *b* are parallel.

The idea is that, such a principle, stipulating that the truth-conditions of its two sides are to coincide, allows us to pass from a prior understanding of the right-hand type of context to a grasp of the type of context on the left. That is undeniable, naturally, if the left-hand side is taken merely as a notational variant, an *unstructured* equivalent of the right. What is distinctive about the Fregean perspective is that we are to take it, in our passage across the biconditional, that the syntax of the left-hand side is *just what it seems*, and that the familiar expressions it contains—in the case of the direction principle, for example, the symbol for identity and the embedded names of lines—are to have their customary meaning. The epistemology of the abstract objects spoken of in contexts of the kind typified on the left-hand side is thus taken to be just that of the presumed (relatively) unproblematic states of affairs depicted by the kind of context featuring on the right.

Is this line so much as coherent? Dummett now seemingly argues that it is not—or at least that it will not combine with any Fregean notion of sense. Sense determines reference. So if the senses of the two sides of, say, the direction principle are the same, they must purport the same references. The Fregean platonist must therefore acknowledge the presence of reference to directions on the right-hand sides of instances of the direction principle, even though his view requires that someone can fully understand such clauses while innocent of the concept of direction. That is to violate the *transparency* of Fregean sense: specifically, the requirement that it must be transparent to one who grasps the sense of an expression what references to what kinds of objects its featuring in (or being itself) a true sentence would involve.[18]

(iii) The third point of attack concerns the 'Caesar Problem'—the indeterminacy which Frege thought afflicted $N^=$ as an account of numerical identity, as manifested in its incapacity to decide statements of the form,

$Nx{:}Fx$ = Julius Caesar,

or more generally, any statement of the form,

$Nx{:}Fx = q$,

where 'q' is a term not given in the form, '$Nx{:}Fx$', nor introduced as an abbreviation for such a term. As noted, it was in effect the Caesar problem that moved Frege to identify the cardinal numbers with extensions.[19] I absolutely agree that, for a variety of reasons,[20] it is essential to the Fregean approach that there be some constructive response to the difficulty. But in Dummett's view, the defence of a Fregean philosophy of arithmetic offered in *Frege's Conception* depended on an inadequate solution to the Caesar problem,—indeed one actually canvassed and rejected by Frege himself[21]—and it would seem that he holds that no clear solution is in view.[22]

(iv) Dummett's scepticism about neo-Fregeanism has its greatest investment, however, in the success of his fourth and main objection—the contention, already noted, that the impredicativity of principles like $N^=$ and Basic Law V renders them impotent to deliver what, in accordance with the Context Principle, is required if the use of the associated singular terms is to be justified thereby, viz. determinate truth-conditions for the statements of identity they serve to introduce.[23]

Why should the impredicativity of $N^=$ be thought to compromise its explanatory power? Dummett's thought here is open to various interpretations, but the most natural—and I think strongest—construal of his objection is the following. Since $N^=$ purports to fix the truth-conditions of statements of numerical identity as those of statements affirming a one-to-one correspondence between concepts, it can provide no more definite an understanding of the former than we can be presumed to possess antecedently to the latter. But an understanding of the latter will depend upon, inter alia, an understanding of the range of the first order quantifiers they involve. Clearly the truth-conditions of, for instance, 'All cattle must be immunised' vary as a function of the range of 'All cattle', with quite different claims being involved depending on whether a tacit restriction to presently living, British cattle is in force, or whether some more generous interpretation is germane. But in the present case—the impredicative first order quantifiers in $N^=$—it is impossible to know what the intended range is without already knowing what cardinal numbers are. So there is a vicious circularity. The intended meaning of instances of the left-hand side of $N^=$ is available only to someone who already knows what is supposed to be explained to him thereby, viz. what numbers are.

Each of these four lines of criticism is expounded with great assurance. I am not convinced by any of them, but treatment in detail is wanted which can have no place in this review.[24] The question for the remainder is rather: how does Dummett believe philosophers of mathematics *should* respond to Frege's legacy, if not in the direction of the platonist logicism, outlined in *Frege's Conception*, that seeks to build on Frege's Theorem?

IV

Dummett's suggestions on this fundamental matter, confined mostly to the concluding chapter, are sketchy and—as I at least found—perplexing. Assume with Dummett that some sort of logicism is to be the preferred account of number theory. A fork immediately looms: is this to be a logicism which takes the apparent ontology of the natural numbers seriously, or is it rather to be a *nominalist logicism*? It seems clear that Dummett will choose the first alternative.[25] He is moreover, as noted, explicit that the Context Principle *is* in principle at the service of an epistemology of abstract objects.[26] Where Frege and the neo-Fregean go wrong, in Dummett's view, is in the *implementation* of that Fregean insight. So both logicism and Context-Principle based platonism are in place in Dummett's vision. How are things to run if not essentially as in *Frege's Conception*?

Any logicist-cum-platonist account of arithmetic has to accomplish three objectives. First, meaning has to be assigned to the statements of number theory in such a way that to follow the assignment through would be to come to understand those statements; the Context Principle will then permit the construal of that understanding as the ability to refer to and quantify over the natural numbers. Second, the assignment of meaning has to be accomplished in such a fashion as to accommodate the utter generality of the applications of those disciplines, the sense in which their fundamental laws are 'laws of thought', applicable to any subject matter—for that is the logicist insight. Finally, the content of the statements of arithmetic must, of course, be so explained as to make it perspicuous that, and why, the fundamental number-theoretic principles are indeed *truths*. How else might all this be accomplished than by the initial assumption of a battery of concepts applicable in any area of discourse—precisely: *logical* concepts followed by the construction on their basis of more or less rigorous contextual explanations of statements expressible in the vocabulary of number theory? It is hard to see how one can follow in Frege's footsteps to the extent that Dummett is apparently prepared to do without following at least that closely.

If that is right, then Dummett's positive endorsement of Frege's basic approach leaves him wanting responses to the very objections which he himself levels against the *Frege's Conception* brand of neo-Fregeanism. First, a solution to the Caesar problem must be found. Dummett, as noted earlier, suggests none.[27] Next, contextual explanations have to be legitimate—because needed—of contexts in which reference to natural numbers is involved. So Dummett wants a response to his own objection to the coherence of this aspect of 'definition by abstraction'. Above all, Dummett has to have some account to offer for how it is possible to do better than the—putatively—viciously impredicative form of contextual explanation embodied in $N^=$ and Basic Law V.

Dummett does attempt to speak to this last need, essaying an application of the idea of *indefinite extensibility* which he first introduced in his paper on Gödel's theorem.[28] The totality of items falling under a given concept, F, is indefinitely extensible if—roughly—for any specification of a definite collection of items comprising only Fs, it is possible to show that there is another item, intuitively also F, which is not included in that collection. *Class* and *ordinal number* are usually offered as paradigmatic indefinitely extensible concepts, since given any definite collection of classes, or ordinal numbers, we can—exploiting the reasoning of Russell's paradox, or the Burali-Forti paradox—immediately generate a new class, or a new ordinal number lying outside that collection. More controversial is Dummett's contention that the fundamental mathematical domains like those composed by the natural numbers, or the real numbers, are likewise indefinitely extensible.[29]

It is a well-known contention of Dummett's that classical logic is inappropriate in the treatment of indefinitely extensible totalities. What is novel here is the suggestion, in the concluding pages of the book, that a proper acknowledgement of the indefinite extensibility of their special domains somehow enhances the prospects of a logicist account of arithmetic and analysis.[30] If the Context Principle alone can provide a workable epistemology of the natural and real numbers, any such logicism must rely heavily on contextual explanations. And these explanations are bound to build on an understanding of certain quantified statements. How do the rules of the game helpfully change if we take it that the domains of these quantifiers are indefinitely extensible?

Suppose *natural number* to be indefinitely extensible, and consider a restriction of $N^=$ to finite concepts. The basic dilemma is that if we insist on a predicative construal of its first order quantifiers, so that natural numbers are excluded from their range, then there is no way of proceeding with Frege's proof of the infinity of the number series. So we will not be able to find a basis for the truth of the fundamental laws of arithmetic in our explanations. But if the quantifiers are taken impredicatively, in the way that proof strategy requires, then, or so Dummett contends, a vicious circle is introduced and we fail genuinely to fix the truth-conditions of statements of numerical identity in the intended fashion. Evidently the thought that *natural number* is indefinitely extensible does nothing to help the logicist play the first horn of the dilemma. So any assistance will have to be in drawing the sting of the second horn.

But how exactly is that supposed to work? True: if the domain of the first-order quantifiers encompasses the numbers, then that domain will now be indefinitely extensible and it will therefore be illegitimate to request a once-and-for-all specification of it. So it will not be appropriate to reproach the Fregean for failing to provide a definite specification of the range of the quantifiers that feature on the right-hand side. But although Dummett does often so express his complaint,[31] the real concern—which alleges an explanatory circle—is neither less nor more impressive than it was before. If, in order to take the statement of $N^=$ in the right way, it is necessary first to know what numbers are and that they are included in the domain of the quantifiers on the right-hand side, then the explanation purportedly offered is compromised by circularity whether or not that domain is understood as an indefinitely extensible one—it seems clear that the indefinite extensibility of the intended domain can do nothing to help deflect the objection.

There is, of course, much more to say. It remains to be seen whether these thoughts of Dummett's can generate a development of Frege's philosophy of arithmetic which genuinely contrasts with those already canvassed by others, and which can respond effectively to the objections—if they are taken to be cogent—levelled by Dummett against *Frege's Conception* and Frege himself. That is perhaps the most intriguing of the questions generated by this masterly but ultimately enigmatic book.[32]

Notes

1. Compare p. 223: "... the error [underlying the inconsistency of *Grundgesetze*] lay much deeper than a misconception concerning the foundations of set theory. It was an error affecting his entire philosophy"; and p. 235.
2. "We have no choice, however, but to treat *Grundlagen* as presenting the greater part of the philosophical underpinnings of the theory of the foundation of arithmetic expounded in *Grundgesetze*, while bearing in mind that, if he had incorporated this material into *Grundgesetze*, he would have subjected it to substantial modification" (pp. 1–2).
3. As Dummett notes (p. 31), by the time of writing *Grundgesetze*, Frege comes to characterise his goal not as a demonstration of the analyticity of arithmetic but of its being a "branch of logic".
4. A developed such account would exploit something like the contrast between 'simple' and 'complex' predicates which is a recurrent notion in *FPL*. It is not clear how the approach might apply, however, to cases where analyticity is recognised non-inferentially but where its recognition is not a precondition of the very understanding of the sentence in question. (Might not somebody understand 'earlier than' to whom it had not occurred that the denoted relation must be transitive?)
5. Of course, there are much stronger grounds, canvassed in, for instance, my *Frege's Conception of Numbers as Objects* (henceforward "*Frege's Conception*") (Aberdeen University Press, 1983) § 6, to which Frege might have had recourse. Dummett reverts to consideration of the motives for rejecting a purely quantificational construction of arithmetic—the "radically adjectival strategy"—in *FPM* Ch. 11, pp. 131–40.
6. Sometimes unhappily called 'definition by abstraction'.
7. The principle, that the number of Fs is identical with the number of Gs just in case the Fs may be put into one-to-one correspondence with the Gs, is attributed

to Hume in *Grundlagen* § 63. Readers of Book 1, part III, section 1 of the *Treatise* may feel Hume is flattered by the attribution. In any case, the title, 'Hume's Principle' has achieved some currency. I referred to the principle as $N^=$ in my *Frege's Conception*. Dummett calls the related principle, which replaces the technical notion of one-to-one correspondence with the informal 'there are exactly as many ...s as ---s', as the "original equivalence".

8. See p. 194 of his 'Frege's Theory of Number' in *Philosophy in America*, ed. Max Black (Allen and Unwin, 1964) pp. 180–203.
9. Dummett (p. 123) affects to regard this demonstration as a waste of time. (Compare the remarks in his article, 'Frege and the Paradox of Analysis' at p. 35–6 of M. Dummett, *Frege and Other Philosophers* (The Clarendon Press, 1991).) But the point is of absolutely fundamental importance for the evaluation of Frege's philosophy of arithmetic; and whereas the reasoning outlined in *Grundlagen* is convincing only if one attempts to think it through in detail after the fashion illustrated in my book, the proofs of the corresponding theorems offered in *Grundgesetze* are littered with abstracts for extensions and courses-of-values, and it is only quite recently that Richard Heck has demonstrated (in his 'The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*', *Journal of Symbolic Logic*, vol. lviii (1993), pp. 579–601) that such reliance on, in effect, the notion of class is a dispensable feature of those proofs.
10. This agreeable honorific was first suggested by George Boolos at p. 268 of his 'The Standard of Equality of Numbers' in Boolos, ed., *Meaning and Method* (Cambridge University Press, 1993), pp. 261–77.
11. More specifically, the system consisting of second order logic with $N^=$ as sole additional axiom is consistent if second order arithmetic is. This has been noted independently by George Boolos, John Burgess and Harold Hodes. For a proof, see Boolos's, 'The Consistency of Frege's *Foundations of Arithmetic*' in Judith Jarvis Thomson (ed.), *On Being and Saying: Essays in honor of Richard Cartwright*, (MIT Press, 1987) pp. 3–20. An improved version of the proof is in the first appendix to Boolos and Heck, 'Die Grundlagen der Arithmetik §§ 82-3', in Matthias Schirn, ed., *Philosophy of Mathematics Today* (The Clarendon Press, forthcoming).
12. The phrase, of course, is that by which Frege several times characterises his project. See, for instance, the Appendix to vol. ii of *Grundgesetze* in which Frege famously reacts to Russell's Paradox. That $N^=$ could in this way play a part in a defensible arithmetical platonism is a key contention of *Frege's Conception of Numbers as Objects*; it is another question whether the objects so sanctioned should be regarded as *logical*, however.
13. Naturally, this contention, even if sustained, would have no immediate bearing on the prospects of extending some form of logicist thesis beyond the arithmetic of finite cardinals. For one in sympathy with the arithmetical claim, one fascinating question will be whether some consistent abstraction principle can be found of the same general type—one deploying a second order equivalence relation on its right hand side—which yields a modified notion of class still strong enough for the purposes of a construction of the theory of the real numbers. For one suggestion in this direction, see the Limitation of Size proposal bruited by Boolos (at p. 231) in his 'Whence the Contradiction?', *Proceedings of the Aristotelian Society*, Suppl. Vol. lxvii (1993), pp. 213–33.
14. See also for instance his *The Interpretation of Frege's Philosophy* (Duckworth, 1981), pp. 424–7 and 452–7.
15. See Chapter 2, section x.

16. It is something of a moot point how close. Dummett writes, somewhat archly (p. 191): "Ultimately, Wright fails to find this intermediate view"—that attributed to Dummett in *Frege's Conception*—"coherent: he doubts if there is any tenable position between the austere"—eliminative reductionist—"and robust"—realist—"interpretations. As concerns contextual definitions, properly so called, I shall here maintain an intermediate view, perhaps one more austere than that which Wright had in mind. I shall however spend no time in discussing either how faithfully Wright represents the views I expressed in *Frege: Philosophy of Language*, or how far those I advance here diverge from them." A pity. Clarity could only have been served if Dummett had taken the trouble to respond to the *Frege's Conception* criticisms, and to define any respects in which the new "intermediate" view represents a change of mind.
17. See pp. 189–99, 234–5, and 239 for development of this line of criticism.
18. For elaboration, see pp. 168–79 and 194–5.
19. —A futile response, of course, even prescinding from the inconsistency of his theory of extensions, since the problem must arise for extensions too.
20. For further discussion of the pressures on the Fregean in this connection, see my 'On the Harmless Impredicativity of $N^=$ (Hume's Principle)' in M. Schirn, ed., *op. cit.* note 11.
21. This is actually a bad misreading of the *Frege's Conception* proposal. For discussion, see Bob Hale's 'Dummett's Critique of Wright's Attempt to Resuscitate Frege' in *Philosophia Mathematica* (3) vol. ii (1994), pp. 122–47.
22. At any rate, none is offered in *FPM*. For elaboration of Dummett's criticisms here, see pp. 155–66 and 213–4.
23. Dummett presents this line of concern repeatedly. See in particular pp. 206–7, 226–7, 232–4, and 236.
24. For penetrating discussion of each of the four lines of objection see Bob Hale's 'Dummett's Critique of Wright's Attempt to Resuscitate Frege' (*op. cit.*, n. 21). The second objection was anticipated and treated in my 'Field and Fregean Platonism' in A. D. Irvine, ed., *Physicalism in Mathematics* (Kluwer, 1990), pp. 73–93, section 4. The fourth objection is discussed in detail in my *op. cit.* note 20 above.
25. Hartry Field, in his *Science Without Numbers* (Blackwell, 1980) and other writings on mathematics, has long been offering what is, in effect, one form of nominalist version of logicism, and Dummett is severely critical of it; see pp. 297–300 and 312–3.
26. See, for instance, pp. 155–6, 181–3, 207–8, 231 and 236.
27. It is true that the exposition offered at pp. 213–4 of Frege's proposed 'solution', in § 10 of *Grundgesetze*, to the special version of the Caesar problem which arises concerning the distinction of the two truth-values from courses-of-values generally is uncritical in tone. However if Dummett means to endorse Frege's proposal, then he ought to have reckoned with the obvious fact that in general the Caesar problem can be solved by stipulation only for ranges of objects which we *already know* are not serious candidates to be numbers, or courses-of-values, or whatever. But *that* is exactly the knowledge which a proper solution ought to be giving us. If we think we have it already, "that is no thanks" to Frege's proposal. Cf. *Frege's Conception*, pp. 112–3.
28. 'The Philosophical Significance of Godel's theorem' in *Ratio* vol. v (1963), pp. 140–55, reprinted in M. Dummett, *Truth and Other Enigmas* (Duckworth, 1978) pp. 186–201.
29. Here no actual contradiction results from the attempt to suppose that there is a definite collection—set—of all natural, or real numbers, and some commentators—

notably Boolos, in his 'Whence the Contradiction?' (*op. cit.* note 13), and Peter Clark, in his contribution to the same symposium and in his 'Dummett's New Argument Against Classical Mathematics', in M. Schirn, ed., (*op. cit.* note 11)—have been critical of Dummett's classification of natural, and real number as cases of indefinite extensibility. On this point, see Dummett's remarks on p. 318 of *FPM* and p. 442 of his 'What is Mathematics About?', in his anthology, *The Seas of Language* (The Clarendon Press, 1993), pp. 429–45.

30. Compare pp. 440 and following of M. Dummett, 'What is Mathematics About?' (*Op. cit.* note 30).
31. For example, at p. 236.
32. Thanks to Peter Clark, Bob Hale and Richard Heck for discussion.

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BOOK REVIEWS

HISTORY OF PHILOSOPHY

Epistemic Logic in the Later Middle Ages
By IVAN BOH
Routledge, 1993. xviii + 190 pp. £37.50

Is it a matter of logic that what is known is true? If so, perhaps it's also a logical truth that all bachelors are unmarried, that 2 and 2 make 4, and that if God tells me something, that is true too.

Hintikka subtitled his book, *Knowledge and Belief*, 'An Introduction to the Logic of the Two Notions'. But epistemic logic (with its twin, doxastic logic) is the Cinderella of applied logic. It goes unrecognised in Gabbay and Guenthner's *Handbook of Philosophical Logic.* Modal logic, deontic logic, even dynamic logic, are now recognised codifications of the logical principles underlying the concepts of alethic modality, obligation and action. Yet many discussions of the factive nature of knowledge, or of the KK-principle, are not articulated as examinations of logical principles. The only systematic survey of epistemic logic, by name and for that matter by deed (in English), seems to be Lenzen's 1978 study in *Acta Philosophica Fennica.*

If there is to be such a creature, what then does epistemic logic look like? This is the heart of the problem: it ranges from issues such as the acceptability of the *KK*-thesis, which at least has the form of a logical principle, that what is known is known to be known (cf. the 4-thesis in modal logic: in symbols it reads $Kp \rightarrow KKp$), via puzzles about *de dicto* and *de re* attitudes (which the mediaevals referred to as compound and divided senses) to straightforward epistemological issues such as externalism vs. internalism (with the latter's