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Author(s): Crispin Wright
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On An Argument On Behalf Of Classical Negation

CRISPIN WRIGHT

I

What characteristics identify the negation of a proposition? It is hardly controversial that nothing can rank as the negation of *A* unless its truth is incompatible with that of *A*. But in his British Academy Lecture,[1] Christopher Peacocke makes two interestingly stronger claims: first, that any ordinary thinker who grasps the concept of negation has the resources to recognise, more specifically, that the negation of *A* is identified as *the weakest proposition* incompatible with *A*; and second, that it follows from this identification that negation sustains its normal *classical* proof theory, in particular the rules of *reductio ad absurdum* (RAA) and double negation elimination (DNE).[2]

Peacocke's second claim is obviously of great importance to the debate about semantical anti-realism and the implications it has usually been thought to carry for logical practice. Peacocke himself stops short of the contention that by showing any alternative to be false to the concept, his argument actually *enforces* a classical treatment of negation. Below (§IV) I shall look briefly at the way he motivates this reserve. But, notwithstanding his own concessive remarks, it would seem that, if Peacocke's argument is good, there can no avoiding the verdict that logics, like that of the Intuitionists, in which DNE is discarded, can only be ill-motivated. Someone who grants the argument must therefore conclude that either the connection, insisted on by Michael Dummett, between semantical anti-realism and revision of classical logic is misconceived, or that semantical anti-realism itself cannot do justice to the conceptual constitution of the logical constants.

II

Someone who wishes to resist Peacocke's argument had better find fault with the transition from the claim (IC), that the negation of *P* is the weakest proposition incompatible with *P*, to the conclusion that DNE is an unrestrictedly valid pattern of inference. For there is—or so it seems to me—no room for dispute of (IC)

[1] Peacocke (1987).

[2] The Law of Excluded Middle will be, of course, an immediate corollary, in view of the straightforward provability of its double negation in any system of sentential logic containing RAA and vel-Intro.

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itself, provided a modest but necessary reformulation is effected. Reflect that talk of *the* weakest statement with such-and-such features is infelicitous in any case where there need be no unique weakest statement with the features in question. So in the present case: we have no general reason to exclude the possibility that there may be a variety of statements—different propositional contents—each incompatible with *A*, each entailed by any statement incompatible with *A*. (IC), formulated so as not to exclude this possibility, will merely affirm that $\neg A$ is true when, but only when something is true which is incompatible with *A*.

Peacocke himself canvasses a number of "essentially equivalent" ways to reach (IC).[3] But the heart of the matter, I think, concerns the equivalence

(N) $\neg A$ is true $\dashv\vDash$ *A* is not true

Reformulated as indicated, it is clear that (IC) is an immediate consequence of (N). For what is certainly true when something incompatible with *A* is true is that *A is not true*—so much is merely enjoined by such statements'incompatibility with A. But then (N) ensures that $\neg A$ is true in the same range of cases, just as reformulated (IC) affirms.

The question, then, is, what if anything imposes (N)? And the answer is that it is a very direct consequence of the familiar equivalence :

(T) $A \dashv\vDash A$ is true,

and the basic and incontrovertible property of negation that the negations of logical equivalents are equivalent. For then we have that

$\neg A \dashv\vDash \neg$|*A* is true|.

Reflecting that substitution of "$\neg A$"for "*A*"in (T) yields

$\neg A \dashv\vDash \neg A$ is true,

we arrive, by transitivity of "$\dashv\vDash$", at

$\neg A$ is true $\dashv\vDash \neg$|*A* is true|.

Since we may take the right hand side of that as a harmless re-write of "*A* is not true", the result is effectively (N).

(IC) is thus a consequence of the feature of the truth-predicate articulated in (T), and must therefore be regarded as uncontroversial on pain of disputing that truth really has that feature.[4]

[3] Three, in fact (Peacocke 1987, p. 163.) One is is to reflect that RAA would not otherwise be valid, since *A*'s generation (on certain assumptions) of incompatible consequences can only require that (on those assumptions) matters stand *somehow* incompatibly with *A*—there could be no valid inference to $\neg A$ if $\neg A$'s truth is merely one determinate (contrast: determinable) way for matters so to stand. A second is to realise that "if $\neg A$ were not the weakest such condition, then there would be a consistent content whose truth requires neither that of *A* nor that of this supposedly stronger negation of *A*". A third is "by starting from the realisation that $\neg A$ is true in any case in which *A* is not". I am not sure what argumentative force attaches to the second. The third is closest to what is crucial, though it needs, I think, the elaboration which follows.

[4] In order to pre-empt one such line of dispute, I have deliberately registered (T) in the form of a claim of mutual logical consequence, rather than the more usual biconditional,

A is true $\leftrightarrow A$.

III

The crux, then, is whether (IC) enjoins DNE. Why does Peacocke regard it as doing so? In his British Academy lecture, he writes, merely, that

> Anything incompatible with $\neg A$, i.e.

—by (IC)—

> something which entails $\neg\neg A$, must also entail A too,—on pain of $\neg A$ not being the weakest condition incompatible with A.[5]

This is somewhat inexplicit. But one concern is salient: the strategy of the train of thought, whatever its detail, would appear to proceed by a *reductio* of the pool of suppositions:

> that P entails $\neg\neg A$;
>
> that P does not entail A;
>
> that (IC) holds.

The most that is immediately in prospect, therefore, would seem to be a demonstration of the *negation* of the second supposition on the assumption of the other two. But if so, it is going to take a DNE step to get the intended result, that—granted (IC)—P entails A whenever it entails $\neg\neg A$. So whatever its detail, isn't the argument going to be circular at best?

True, the needed presupposition is not of the validity of DNE quite generally, but only of its validity as applied to statements of the form,

> $\neg\neg \ulcorner P$ entails $A \urcorner$.

Still, an argument would be owing why that limited case should have some prior claim to acceptability.[6]

This misgiving is reinforced when we consider a second, much more explicit presentation which Peacocke offers in his paper "Proof and Truth".[7] Let "A/B" express that A is incompatible with B. Peacocke writes:

> If **O** is the weakest operator for which $P/\mathbf{O}P$ holds generally, the classical introduction and elimination rules are valid for it. ... suppose we had a case in which this inference fails

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The biconditional is, of course, going to be suspect, on natural readings of "$\leftrightarrow$", for anyone who is working with truth-values besides truth and falsity, or with truth-value gaps—and who, as is natural, takes "A is true" as *false* when A has a third, or no truth-value. The claim of mutual logical consequence, by contrast, is answerable only to what valuation is required, as a matter of logical necessity, whenever either ingredient takes the value, *true*.

[5] Peacocke (1987 p. 164).

[6] Intuitionistically, DNE is, of course, uncontroversial when applied to sequents whose conclusions are effectively decidable. But of course entailment statements are not so, even if we restrict our attention to those whose ingredients are expressible entirely in first-order terms. In general, it is obscure what motive there might be to accept DNE for entailment statements as a class which was not a motive to accept it without restriction.

[7] Peacocke (forthcoming).

$$\frac{\mathbf{OO}A}{A}$$

In this case we would have that **OO**A is true and A not true. Since **OO**A is incompatible with **O**A, **O**A must be not true in the given case. But if A and **O**A are both not true, **O**A is not the weakest proposition incompatible with A, for being the weakest entails that **O**A is true in any case in which A is not. Since by hypothesis **O**A is the weakest such proposition, there can be no case in which the analogue of classical negation elimination fails.[8]

A reaction to this corresponding to the reservation just expressed would be that no cogent argument for the validity of the classical negation rules as applied to the operator, **O**, can proceed by *reductio* on the premise-set, {**OO**A, A is not true} —at least, not if RAA is taken in the form:

$$\frac{\begin{array}{cc}\Gamma & \Gamma \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ B & \neg B\end{array}}{\neg\{\Gamma\}}$$

For all that will follow from a proof that not all the members of {**OO**A, A is not true} are true is the conditional,

$$\mathbf{OO}A \rightarrow \neg(A \text{ is not true}),$$

rather than the sought-after

$$\mathbf{OO}A \rightarrow A;$$

and it needs DNE to get us from the former to the latter.

Now, this reaction is not quite technically fair, since Peacocke's argument does not (quite) put up {**OO**A, A is not true} for *reductio*, but rather something of which the possible truth of that premise-set is an immediate consequence, namely the supposition of a failure of the inference pattern,

$$\frac{\mathbf{OO}A}{A.}$$

But the basic difficulty remains. All that can follow from a *reductio* of that supposition via the RAA rule schematised is that there is no A for which **OO**A does not entail A. Write "ΦA"to indicate that there is an entailment from **OO**A to A. Then the conclusion of the argument will be a claim of the form, $\neg(\exists A)\neg\Phi A$. We can certainly advance from that to a claim of the form, $(\forall A)\neg\neg\Phi A$. But the sought-after claim is that DNE is unrestrictedly valid, i.e., $(\forall A)\Phi A$. And to get to that, we need, once again that DNE hold good, if not unrestrictedly, then at least for entailment-statements.

There would not be this difficulty, of course, if, rather than the version schematised above, we had RAA in the classical form:

[8] Peacocke (forthcoming).

$$\frac{\begin{array}{cc} \neg\{\Gamma\} & \neg\{\Gamma\} \\ \vdots & \vdots \\ B & \neg B \end{array}}{\Gamma}$$

when Γ could be taken as {**OO**A entails A} and the reasoning could proceed, via $\Diamond${**OO**A, A is not true}, exactly as Peacocke describes. But this version of RAA is just a convenient way of packaging standard RAA plus DNE into one rule—it can hardly be clear that there is a motivation for it based on (IC) or other uncontroversial properties of negation if we have yet to descry a clear motive for DNE itself.

IV

Putting on one side, for a moment, these reservations about its cogency, there is a puzzling aspect to Peacocke's commentary on his argument. He writes:

> It may be objected that an intuitionist will equally agree that it is primitively obvious that $\neg A$ is incompatible with A. Indeed he may also agree that there is nothing weaker than $\neg A$ which is incompatible with A. But if this is so, how can these two points determine classical rather than intuitionistic negation?
>
> The answer is that it is not the same points which hold for the intuitionist as for the classicist. When the intuitionist agrees that it is primitively obvious that $\neg A$ is incompatible with A, what he means by "incompatible" is not what the classicist means. What the intuitionist means by the incompatibility of A with B is that the supposition that A and B are both verified leads to absurdity. In the intuitionist's sense, A is incompatible with "It is not verified that A". That is congenial to him, but there is no incompatibility on the classicist's realistic notion. Since different notions of incompatibility are being used, there is no sound objection to the claim that the semantic value of classical negation is determined.[9]

The puzzle emerges if we refer to the argument from "Proof and Truth" and ask the simple question: at what point in the argument is it assumed, tacitly or otherwise, that the notion of incompatibility is being taken in a realist (non-constructive) sense? The answer, so far as I can see, is: nowhere. To simplify matters, let the supposition for *reductio* be the negation not of an entailment claim but of the corresponding conditional,

OOA is true $\rightarrow$ A is true.

[9] Peacocke (1987, p. 165).

The intuitionist ought—in this particular case[10]—to have no objection to the transition from the negation of that to

$$\mathbf{OO}A \text{ is true} \wedge A \text{ is not true.}$$

and when all relevant notions are interpreted constructively, "**OO***A* is true"and "*A* is not true"are contradictory for the intuitionist no less than for the classicist. Their intuitionistic conjunction affirms that each can be proved. And to prove that *A* is not true is to prove that no constructive verification of *A* can be achieved—that any such verification could be used to generate a proof of a contradiction. Since that is exactly what is affirmed by the intuitionistic negation of *A*, as standardly interpreted, the Intuitionist will have no objection to the equivalence (N) nor, in consequence, to the *reductio* Peacocke outlines in the second half of the passage quoted. What the intuitionist *will* object to is taking that reasoning as a proof, assuming (N) and that **OO***A* is true, of the truth of *A*, rather than of the untruth of "*A* is not true". *This* objection, however, has nothing to do with a constructive interpretation of incompatibility.

It is true that Peacocke's last quoted remarks refer to the terse statement in the British Academy lecture. But we are in no position to contrast the argument to which that statement refers with the "Proof and Truth" version. And a similar invocation of the contrast between the classical and intuitionist notions of incompatibility also occurs in "Proof and Truth".

V

Can Peacocke's argument from (IC) to DNE be more convincingly reconstructed? We noted, in effect, that no convincing version of an argument from (IC) to DNE can proceed by *reductio* if the standard version of RAA is all we may assume; for then the reasoning will need in addition to rely on a DNE step. Since it does not seem as if the classical version of RAA could be motivated independently of some explicit argument for DNE, it would seem that a convincing version of Peacocke's thought cannot proceed by *reductio* at all.

How else might it proceed? Well, there is, I think, another line which may be read into the terse statement in the British Academy lecture. Write $\mathbf{W}(A, B)$ for: *B* is a weakest statement incompatible with *A*.

Plainly, if $\mathbf{W}(A, B)$, then any statement which entails that *A* is not true, and thereby via (IC) entails $\neg A$, entails *B*. So in particular, $\neg A$ itself entails *B*. Now suppose the relation, $\mathbf{W}(A, B)$, is symmetric. Then whenever $\mathbf{W}(A, B)$ holds, so does $\mathbf{W}(B, A)$, and hence, by the reflection just mooted, $\neg B$ entails *A*. Since (IC) ensures that $\mathbf{W}(A, \neg A)$, it follows—taking $\neg A$ for "*B*"—that $\neg\neg A$ entails *A*. QED.

The reasoning may avoid all use of *reductio*, then, if we may assume the symmetry of "**W**"—assume that whenever *B* is a weakest statement incompatible

[10] Of course, he will not accept the transition from $\neg(A \rightarrow B)$ to $A \wedge \neg B$ in general.

with A, A must likewise be a weakest statement incompatible with B. And such an assumption may seem quite intuitive—at least to a classical outlook which reasons as follows. If B is a weakest statement incompatible with A, it is true in *all* cases when A fails of truth—so if B fails of truth, it cannot be one of *those* cases with which we are concerned. But the only other kind of case are ones where A is true: so any statement which entails that B fails of truth must entail A.

However, a less than classical outlook will find cause for concern in the implicit transition from the thought that the hypothesised case is not one in which A fails of truth to the thought that it must be a case in which A is true! In fact, the supposition that "**W**"is symmetric is quite trivially equivalent to the supposition that DNE holds good. Perhaps the simplest way to see the point is to reflect that $\mathbf{W}(A, B)$ is tantamount to the claim that B mutually entails $\neg A$.[11] So the symmetry of "**W**"is just the thesis of the equivalence of $\neg A \dashv\vDash B$ with $\neg B \dashv\vDash A$. And that equivalence entails and, in the company of otherwise uncontroversial rules, is entailed by DNE.[12]

That does not yet show that to argue from the symmetry of "**W**"to the validity of DNE is merely to skirmish in a circle. The question is whether an *independent* ground can be supplied for the symmetry of "**W**"—a ground independent, that is, of classical sympathies. But there is every reason to doubt it. Reflect that the question, which statements are incompatible with a given statement and, a fortiori, which statements are weakest such statements, has a certain relativity—it depends on the universe of statements presupposed. And the fact is that the classicist and the constructivist (intuitionist) are not in general agreement about what statements—intelligible thought-contents—there are. For the intuitionist, for instance, the universe of intelligible mathematical thoughts consists—very crudely—in nothing but contents for which there is no difference between the claim that P and the claim that it is intuitionistically provable (I-provable) that P. What, in such a universe, is a weakest statement incompatible with P? Well, it is a weakest statement incompatible with the I-provability of P, so it is equivalent to the claim that P is not I-provable, i.e. to the claim that it is I-provable that it is not I-provable that P. So a weakest statement incompatible with that in turn will be equivalent to the claim that it is not I-provable that it is not I-provable that P, i.e. to the claim

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But the material character of intuitionistic "$\rightarrow$"ensures that the intuitionist will be content with the transition from $\neg(A \rightarrow B)$ to $\neg\neg A \wedge \neg B$. So—bearing in mind that *triple* negation does collapse, intuitionistically, into single negation,—it follows that the stronger transition does go through if A is already a double negation, or equivalent to such, as in the present case.

[11] $\mathbf{W}(A, B)$ says two things: that B is incompatible with A, and that any statement incompatible with A entails B. The first is equivalent to the claim that B entails $\neg A$; the second to the claim that $\neg A$ entails B (since P entails Q just in case any R which entails P entails Q).

[12] That $A \vDash \neg B$ can be inferred from $\neg A \dashv\vDash B$ using standard RAA; and that $\neg B \vDash A$ can be inferred from $\neg A \dashv\vDash B$ if we can eliminate the double negation in $\neg B \vDash \neg\neg A$. Conversely, if we are given the equivalence of $\neg A \dashv\vDash$ B with $\neg B \dashv\vDash A$, we may easily pass from $\neg\neg A$ to $\neg B$ via the first, and then from $\neg B$ to A via the second.

that it is I-provable that it is not I-provable that it is not I-provable that *P*. But the latter, in any case where *P* is not effectively decidable, is a potentially weaker statement than the claim that it is I-provable that *P*, i.e. the claim that *P*. For it may be that we can establish (I-prove) that there is no establishing that *P* cannot be I-proved without thereby accomplishing an I-proof of *P*. It follows that "**W**"cannot be guaranteed symmetric in the universe of intuitionistic thoughts.

The point is good for any domain of thoughts where three conditions are met:

(i) truth within the domain is conceived as biconditionally dependent on the satisfaction of some epistemic constraint—call it "verifiability";

(ii) the domain is closed under negation;

and

(iii) we cannot guarantee that for any thought, *P*, in the domain, either *P* or its negation is verifiable,

"**W**"cannot be guaranteed symmetric across such a domain of thoughts. For since, by (iii), we lack reason to suppose of an arbitrary *P* in the domain that either it or its negation is verifiable, it cannot in general be that to verify that the negation of *P* is not verifiable is verify *P*. But (i) and (ii) ensure that to verify that the negation of *P* is not verifiable is effectively to verify the double negation of *P*—a weakest proposition incompatible with not-*P*. So *P* cannot also be a weakest proposition incompatible with not-*P*, even though not-*P* is a weakest proposition incompatible with *it*.

We should conclude that always provided a domain of thoughts may indeed meet the three conditions stated, Peacocke is mistaken in supposing that our most basic understanding of negation, as incorporated in (IC), provides any push in the direction of a distinctively classical conception of that connective. Indeed, in a sense there *is* no distinctively classical conception of negation and no distinctively classical conception of incompatibility either. Both intuitionist and classicist can adhere to the conception of negation characterised by (IC):

$$\neg A \text{ is true if and only if some } B \text{ is true such that } \mathbf{W}(A, B),$$

and agree in their understanding of "W". The right account of their disagreement is that it concerns what distinctions may intelligibly be made among the propositional contents falling within the range of these operators—whether, for instance, Goldbach's conjecture may be distinguished from the proposition that it is constructively provable that every even number is the sum of two primes. All their other, more local differences flow from the application of this one basic dispute to concepts in which they are otherwise agreed. If this is correct, any argument which, like Peacocke's, purports to show that a classical conception of this or that connective is enforced by more local considerations will, if valid, rest on assumptions which surreptitiously beg the question against an intuitionistic universe of thoughts.

Universities of St. Andrews
and Michigan, Ann Arbor

CRISPIN WRIGHT

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