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IS HIGHER ORDER VAGUENESS COHERENT?

By Crispin Wright

T

IT is widely assumed not merely that the Sorites afflicts vague expressions only, but that it is a paradox of vagueness — that vagueness is what gives rise to it. Since almost all expressions in typical natural languages are vague, that belief brings one uncomfortably close to the thought, advocated by such philosophers as Peter Unger,¹ that natural languages, and the conceptual systems which they embody, are typically incoherent. But I think the thought that vagueness, per se, generates Sorites-susceptibility is a muddled thought.²

When spelled out it goes, presumably, something like this. If F is vague, its very vagueness must entail that in a series of appropriately gradually changing objects, F at one end but not at the other, there will be no nth element which is F while the n+1st is not; for if there were, the cut-off between F and not-F would be sharp, contrary to hypothesis. Accordingly, the vagueness of F over such a series must always be reflected in a truth of the form:

(i) $\neg(\exists x)(Fx \& \neg Fx')$, (where x' is the immediate successor of x.)

That, of course, is a classical equivalent of the universally quantified conditional which is the major premiss in standard formulations of the Sorites — a thought which prompted Hilary Putnam to suggest that a shift to intuitionist logic might be of value in the treatment of vagueness. So indeed it might, in other connections. But it won't help here; for intuitionistic logic will yield a paradox from (i) (as will any logic with the standard ∃ and &-Introduction rules + reductio ad absurdum).³ This form of the paradox — the No

See for instance Unger [4].

² And that the most convincing and troublesome versions of the Sorites draw on *other* features of the (vague) expressions they concern. See sections I and VII of my [5].

<sup>[5].

3</sup> Of course, what intuitionistic logic does make possible is a response to the paradox which treats it as a reductio of the major premiss, hence as a proof that ¬¬(∃x)(Fx & ¬Fx'), without sustaining the double negation elimination step which forces that conclusion into a statement the precision of F. But the basic difficulty remains: whether or not the double negation is sustained, to treat the paradox as a reductio is to deny a premiss which seems to say merely that F is vague. Any genuine solution to the paradox has therefore to explain how that appearance is illusory — how the major premiss fails as a schematic description of vagueness. There is no way around the obligation and no reason to think, once it is met, that any further purpose will be served by imposing intuitionistic restrictions in this context.

Sharp Boundaries paradox — thus appears to constitute a proof that (one kind of) vagueness is eo ipso a form of semantic incoherence.

Only 'appears' though. What, it seems to me, the No Sharp Boundaries paradox brings out is that, when dealing with vague expressions, it is essential to have the expressive resources afforded by an operator expressing definiteness or determinacy. Someone might think that the introduction of such an operator can serve no point since there is no apparent way whereby a statement could be true without being definitely so. That is undeniable, but it is only to say that - in terms of a distinction of Michael Dummett ([1], pp. 446-7) — the content senses of 'P' and 'Definitely P' coincide; whereas the important thing, for our purposes, is that their ingredient senses — their contribution to contexts embedding them — differ, the vital difference concerning the behaviour of the two statement-forms when embedded in negation. Equipped with an appropriate such operator, we can see that a proper expression of the vagueness of F with respet to the relevant sort of series of objects is not provided by (i) but rather requires a statement to the effect that no definitely F element is immediately succeeded by one which is definitely not F; that is

(ii)
$$\neg(\exists x)[Def(Fx) \& Def(\neg Fx')]$$

And this principle generates no paradox. The worst we can get from it, with or without classical logic, is the means for proving, for successive x', that

(iii)
$$Def(\neg Fx') \rightarrow \neg Def(Fx)$$
.

Nothing untoward follows from that.

II

If, however, we take seriously the idea of higher order vagueness, then a case can be made that this merely postpones the difficulty. For if the distinction between things which are F and borderline cases of F is itself vague, then assent to

(iv)
$$\neg (\exists x) [Def(Fx) \& \neg Def(Fx')]$$

would seem to be compelled even if assent to (i) is not. So once again the materials for paradox seem to be at hand, each ingredient move taking the form of a transition from $\neg Def(Fx')$ to $\neg Def(Fx)$.

But the following is the obvious reply. Of any pair of concepts, F and H, which share a blurred boundary, we shall want to affirm

(v)
$$\neg(\exists x)[Def(Fx) \& Def(Hx')]$$

when x ranges over the elements of an appropriate series in which the blurred boundary between F and H is crossed. The original problem occurred in (i) when, with $\neg F$ in place of H, we overlooked the need to prefix the predicates with a definiteness operator. And now we are guilty of the same oversight again in (iv); it is merely that this time H has been replaced by $\neg Def(F)$. As soon as the inclusion of the definiteness operator is insisted on, all that emerges is

(vi)
$$\neg(\exists x)[Def(Def(Fx)) \& Def(\neg Def(Fx'))]$$

which yields nothing more than the harmless

(vii)
$$Def(\neg Def(Fx') \rightarrow \neg Def(Def(Fx))$$

Evidently the trick will generalize; so we need never, it seems, be at a loss for a way of formulating F's possession of vagueness, of whatever order, in a way that avoids paradox.

Ш

But this is too quick. It is possible to be confident that the sort of formulation illustrated by (vii) avoids paradox only because we have so far no semantics for the definiteness operator, and are treating it as logically inert. Without considering in detail what form a semantics for it might take, a crucial question is whether it would be correct to require validation for this principle:

(DEF)
$$\frac{A_1 \dots A_n \models P}{A_1 \dots A_n \models Def(P)}$$

provided $\{A_1 \dots A_n\}$ consists of propositions all of which are 'definitized'.

For, in the presence of DEF, and assuming that the corrected formulation, (vi) above, of what it is for the borderline between F and its first-order borderline cases to be itself blurred, is itself *definitely* correct, the harmless (vii) gives way to

(viii)
$$Def(\neg Def(Fx')) \rightarrow Def(\neg Def(Fx)),$$

whose generalization will enable us to prove that F has no definite instances if it has definite borderline cases of the first order. By

⁴ Proof:		
1	(1) $Def \neg (\exists x)[Def(Def(Fx)) \& Def(\neg Def(Fx'))]$	Ass.
2	(2) $Def(\neg Def(\mathbf{F}\mathbf{x}'))$	Ass.
3	(3) $Def(Fx)$	Ass.
3	(4) $Def(Def(Fx))$	3, DEF.
2,3	(5) $(\exists x)[Def(Def(Fx)) \& Def(\neg Def(Fx'))]$	2,4, ∃-intro.
1	(6) $\neg (\exists x)[Def(Def(Fx)) \& Def(\neg Def(Fx'))]$	1, Def-elim.
1,2	$(7) \neg Def(\mathbf{Fx})$	3,5,6,RAA.
1,2	(8) $Def(\neg Def(Fx))$	7,DEF.
1	$(9) \ \textit{Def}(\neg \textit{Def}(Fx')) \rightarrow (\textit{Def}(\neg \text{Def}(Fx)))$	2,8CP.

contrast, (iii) gives way, by parallel reasoning, only to the innocuous

(ix)
$$Def(\neg Fx') \rightarrow Def(\neg Def(Fx))$$
.

The trouble is thus distinctively at higher-order.

DEF says, in effect, that the truth of each of a set of fully definitized propositions ensures that every consequence of that set is likewise definitely true. This may get some spurious plausibility from conflation with the distinct and indisputable principle that whatever is a consequence of a set of propositions each of which is definitely true is itself definitely true. But DEF is plausible in any case. In effect it comes to the claim that when a proposition of the form: it is definitely the case that P, is true, it cannot be less than definitely true. If DEF is valid, and (vi) is a satisfactory characterization, then higher-order vagueness — always a difficult and vertiginous-seeing idea — would seemingly be an intrinsically paradox-generating phenomenon (ergo, presumably, a delusory one).

IV

Interesting recent work of Mark Sainsbury's raises a number of points bearing on this paradox.⁵ I shall comment on three aspects of his discussion.

First, Sainsbury objects that in (vi), which he takes as its classical equivalent

$$(vi)^c$$
 $Def(Def(Fx)) \rightarrow \neg Def(\neg Def(Fx')),$

I picked a needlessly vulnerable characteristic sentence for higher order vagueness. The motivation for (vi), recall, was that if x' is a borderline case of F, it will at least be true that it is not definitely F; and that if it is a *definite* borderline case, then the same will be definitely true. Thus (vi) or, if you will, (vi)^c says that no (definitely) definite F thing is succeeded by a definite borderline case — that the distinction between the Fs and the definite borderline cases is not one with an abrupt threshold, not a sharp one. Isn't that just what second-order vagueness ought to be?⁶

⁵Mark Sainsbury [2]. Sainsbury's paper is a reaction to my [5], in which the No Sharp Boundaries paradox for higher-order vagueness was first, to the best of my knowledge, presented. See also Sainsbury's [3].

knowledge, presented. See also Sainsbury's [3].

⁶There is a slight infelicity here, in so far as '¬Def(Fx)' is not actually definitive of X's being a borderline case of F, but will also be true if x is a negative case. But nothing important hangs on this. The thought in the text is restored — if good at all – by restricting the range of 'x' to positive and borderline cases of F. Alternatively, the reader may prefer to treat '¬Def(Fx)' as characterizing the agglomerate of borderline and negative cases together: (vi) and (vi)' then plausibly capture what it is for the distinction between F's and this agglomerate to be vague — which is just what it is for F to be second-order vague.

Sainsbury ([2], p. 176) believes that there are other equally well motivated candidates for the characteristic sentence, from which one cannot

by a proof of the same general structure as Wright's, derive anything paradoxical... So their entitlement to represent vagueness needs to be undermined before any conclusion antithetical to vagueness can be drawn from Wright's proof.

What candidates? Well, it is natural, when vagueness is at issue, to want to work with some notion, however intuitive, of truth-value gaps, or of truth values other than truth and falsity. Either will set up the possibility of a distinction within the notion of negation. One notion, the *proper* negation of A, may be defined as true if A is false, and false if A is true. The *broad* negation of A, by contrast, while false if A is true, will be true in any other case — true just in case A is other than true. In these terms, it is natural to characterize a borderline case of F as something such that neither the claim that it is F nor the proper negation of that claim is true. Writing 'Neg A' for the proper negation of A and 'Not A' for the broad, x is thus a borderline case of F if

Not Fx & NotNeg Fx

This gives us something else to play the role of ' $\neg Def(Fx')$ ' in (vi)'. Reflecting that no intuition is offended by replacing the occurrence of 'Def(Def(Fx))' by one of 'Def(Fx))', we arrive at Sainsbury's first alternative:

(vi)*
$$Def(Fx) \rightarrow Not Def(Not Fx' & NotNeg Fx')$$
.

Alternatively, suppose we stipulate, as is quite natural, that the claim that A is definitely true, while true when A is true, ranks as false in all cases save where A is true. So 'Def(Fx')' is false when x' is borderline for F. Then

will be true of such an x'. The thought that no x', next to a definite F, is a borderline case of F, can then be expressed by

$$(vi)^{ss}$$
 $Def(Fx) \rightarrow Not Neg Def(Fx').^7$

Sainsbury's claim is that neither (vi)s nor (vi)s generates a paradox

where 'Def'' expressed n iterations of 'Def'. As he remarks, this works out only for n > 0 — hence his shift to (vi)s. Note that the latter shares the harmless infelicity of (vi) and (vi)c — see note 6 above.

⁷Sainsbury does not quite motivate (vi)** this way. I have simplified his discussion slightly, which had a generalized version of (vi)* characterizing vagueness of order n+1 as

 $Def''(Fx) \rightarrow Not Def''(Not Fx' & Not Neg Fx'),$

in the fashion of (vi) and (vi)°; but that their credentials as characteristic sentences for higher order vagueness are no less plausible.

But cannot we get paradox with (vi)* and (vi)**? It is a consequence of the stipulation by which the latter was motivated that no provision is made for the untruth of Def(A) save by its being false. So there is no contrast between NegDef(Fx') and NotDef(Fx'), and (vi)*s is equivalent to:

$$Def(Fx) \rightarrow NotNotDef(Fx')$$
.

Since reductio ad absurdum is presumably valid in the form:

$$\frac{A_1 \dots A_n \models B \& NotB}{A_1 \dots A_{n-1} \models NotA_n}$$

that is going to ensure the availability of

Not
$$Def(Fx') \rightarrow NotDef(Fx)$$
,

and hence reintroduce paradox.8

The situation with (vi)'s is slightly more complicated. Presumably we will want to sustain

(x)
$$Def(Fx) \rightarrow NotNegFx'$$
,

which says, merely, that each definite F is immediately succeeded by something which may not truly be denied to be F and ought surely to hold in any series of the sort which concerns us. But if (x) is a definite truth, then, in the presence of (DEF), we may strengthen it to

(xi)
$$Def(Fx) \rightarrow Def(NotNegFx')$$
,

which, in company with (vi)s, ensures that

(xii)
$$Def(Fx) \rightarrow Not Def(Not Fx')$$

holds. Now (xii) must not be confused with (iii), whose contrapositive it resembles. For (xii), unlike (iii), is formulated in terms of broad negation and it is very unclear, in consequence, whether, like (iii), it may be regarded as harmless. To see the difficulty, reflect that, whatever we stipulate about the semantic value of Def(A) when A is borderline, there is a powerful intuition that its value ought to coincide with that of A when A is polar — is true or false. But broad negation was characterized as false when A is true and true in any other case, including any form of indeterminacy. So it is quite unclear how the broad negation of A can be anything but polar. Hence the semantic value of Def(NotFx') cannot diverge

$$\frac{A_1 \dots A_n \models \text{Not } Def(A \& B); A_1 \dots A_n \models Def(B)}{A_1 \dots A_n \models \text{Not } Def(A)}$$

⁸ Sainsbury in effect remarks on this.

⁹The principle here appealed to is

from that of NotFx', so the occurrence of 'Def' in the consequence of (xii) is idle, and there is no evident way of resisting its conversion to

(xiii)
$$Def(Fx) \rightarrow NotNotFx'$$
.

Since there is no evident objection to Double Negation Elimination for 'Not', 10 that yields

(xiv)
$$Def(Fx) \rightarrow Fx'$$
.

And while (xiv) won't formally yield a Sorites, all that then stands in the way of the paradox is a failure to exploit the powerful intuition canvassed above, that the semantic value of Def(A) should coincide with that of A whenever A is polar, and hence that Def(Fx') should be true when Fx' is — that is, by (xiv), whenever Def(Fx) is.

It is far from clear, then, that either of Sainsbury's proposed alternatives, (vi)s and (vi)s, contributes to any case at all that the apparent link between higher order vagueness and paradox is merely an artefact of reliance on (vi) or (vi) as characteristic sentences. Moreover there is a serious doubt whether anything better could be forthcoming in the framework he favours. I suggested above that, when dealing with vague expressions, it is essential to have the expressive resources afforded by Def. One moral of the foregoing, I suspect, is that it is essential to lack the expressive resources of the sort of broad negation operator produced by Sainsbury - an operator which always generates a polar sentence, no matter what the status of the sentence it operates on. That is the feature which gives rise to the eliminability of double broad negations (see note 10). But the basic natural deduction proof of the double negation of the law of excluded middle uses only rules which seem perfectly unexceptionable where broad negation is concerned. 11 So there is no obstacle to the law itself in the form

A or Not A.

¹⁰ Suppose A is something other than true. Then Not A is true; so, by the matrix for 'Not', NotNot A is false - so of truth-value equal or inferior to that of A. Hence

 $NotNotA \rightarrow A$

will be acceptable on the standard sort of many-valued semantics for '→'. (Double Negation *Introduction*, by contrast, will fail for broad negation, as the reader will swiftly verify.)

11 Viz. reductio ad absurdum, in the form remarked on above, and vel-introduction.

ıuə		
1	(1) Not(P v NotP)	Ass
2	(2) P	Ass
2	(3) P v NotP	2,vel-intro.
2	(4) NotP	2,3,1,RAA
1	(5) P v NotP	4.vel-intro.
	(6) NotNot(P v NotP)	1.5.1.RAA

And its arrival marks the demise of any hope of a satisfactory characterization of higher-order vagueness. With the law in place, we are powerless to refuse the claim that, for each Def(Fx'), either it or it broad negation will hold. The cost of blocking the Sorites which would result if we were to accept each Def(Fx') is thus that we must at some point broadly deny one — tertium non datur. And as soon as we do, we have established a sharp boundary to definite F-ness, when the whole point was to provide an apparatus to describe what is involved in there being none.

Prescinding from the detail of Sainsbury's proposals, there is, in any case, a more basic question about his strategy. Recall that the purpose in introducing (vi)st and (vi)st was to provide at least prima facie coherent characteristic sentences for higher-order vague

predicates whose

entitlement to represent vagueness needs to be undermined before any conclusion antithetical to vagueness can be drawn from Wright's proof.

Surely this thought has matters precisely backwards. It is rather the entitlement of (vi), or (vi)^c to represent vagueness which needs to be undermined before any comforting conclusion about the status of vagueness could be drawn from the availability—if any are available—of paradox-free characterizations. If a notion has an intuitively acceptable characterization which generates paradox, it is not progress towards a resolution of matters merely to devise other seemingly acceptable characterizations which, so far as one can see, avoid paradox. So long as nothing is done to disarm the intuitive credentials of the villain, they—the credentials—merely transfer into grounds for thinking that the apparently innocent characterizations either fail to do justice to the intended notion or are not really innocent.

V

Sainsbury's second main reservation about the paradox concerns the role of the inference rule, DEF. He writes ([2], pp. 175-6):

... one cannot feel happy with the introduction of the undefined 'Def' followed immediately by an assumption about its logic which leads to paradox. It would seem a clear possibility that there should be a conception of 'Def' upon which it demands progressively higher standards. Such a conception would fail to validate... the definitization rule, DEF, and would need to be argued against if higher order vagueness is to be shown paradoxical by the argument.

These are sensible reactions, but they betray some misunderstanding of my original discussion. One thing I regard as definite progress, in an area where it is exceedingly hard to make any, is the modest insight that the No Sharp Boundaries paradox may be

defused by appropriate use of an operator of definiteness. What I sought to show was that that point, which ought to extend to a coherent characterization of vagueness of higher order if that notion is coherent at all, will not so extend unless DEF fails in some relevant way. The alternatives are thus to find fault with (vi), to disclose a relevant failing in DEF or face the consequence that higher order vagueness is *perse* paradoxical. Sainsbury writes as if I had claimed to establish the third disjunct. By my aim was the disjunction.

Nevertheless, he seems to underestimate the problem of disclosing a relevant failing in DEF. It is perfectly true that if, as Sainsbury puts it, 'Def' demands 'progressively higher standards' — if, in other words, in order for Def(A) to reach a certain level of acceptability, A must in general surpass it — then DEF will be invalid for a range of cases in which not all of the premisses in a sequent to which it is to be applied are polar. To take the simplest case: if, when Def(A) is non-polar, Def(Def(A)) drops lower in degree of acceptability, then — since valid inference ought to be degree-of-acceptability preserving — the sequent

$$Def(A) \models Def(Def(A))$$

will be invalid. But to advance that reflection is not yet to find fault with the kind of application of DEF essentially involved in the proof of the paradox. Consider the application at line 8 (see note 4). The objection is in effect that DEF may fail for cases in which the conclusion of its premiss-sequent was not polar, since the definitization of that conclusion might drop crucially further in truth-value, as it were. But such a case can arise only if (some of) the assumptions of the premiss-sequent are already themselves non-polar — otherwise the premiss-sequent would not be a valid entailment in the first place. And in that case the objection is beside the point. For assumption 1, we are taking it, is true - somuch is the force of any assumption; and we presumably so select our starting point for the relevant Sorites that assumption 2 is likewise true. But if 1 and 2 are true claims, then so is line (vii). Similarly, at line (iv), DEF is used to obtain a conclusion from an assumption -3 - presumed true. So if either of the uses made of DEF in the proof is to fail, definitization must be capable of producing a drop in truth-value even when applied to true premisses. Sainsbury does nothing to motivate that idea, and it is not clear how it might be motivated. For it is just the denial of the 'powerful intuition' bruited earlier in discussion of (vi).

VI

But Sainsbury's most far-reaching and distinctive contention ([2], p. 170) is that it is in any case a mistake to try to capture vague-

ness, of whatever 'order', by means of a characteristic sentence -

... a sentence schema, containing a schematic predicate position, such that the sentence resulting by replacing the schematic element by a predicate is true iff that substitute is a vague predicate.

The impression to the contrary derives, in Sainsbury's view, from the lingering influence of what he styles the 'classical conception' of vagueness: the idea that what defines a vague predicate is that it effects a tripartite division — into positive, negative and borderline cases, respectively — where a precise predicate determines a merely bipartite one. The difficulty is then to say something coherent about how the tripartite division can itself be blurred at the edges — so that a merely tripartite distinction is seemingly not enough. My offering of

(vi)
$$\neg (\exists x)[Def(Def(Fx)) \& Def(\neg Def(Fx'))]$$

was exactly an attempt to produce the execrated sort of characteristic sentence: it tries to say what it is for the distinction between the definite Fs and the definite borderline cases to be itself vague — for the transition between the two kinds of case not to occur at an abrupt threshold. According to Sainsbury, this attempt is misguided in principle. Rather we should recognize ([2], pp. 179-80) that

The right way to characterize the vagueness of a predicate is by the fact that it classifies without drawing boundaries: it is boundaryless. A boundaryless predicate allows for borderline cases, but this is not its defining feature. A boundaryless predicate draws no boundary between its positive and negative cases, between its positive cases and its borderline cases, between its positive cases and those which are borderline cases of borderline cases. The phenomena which, from a classical viewpoint, lead to notions of 'higher order vagueness' are accounted for by boundarylessness... To convince you that boundaryless classification is possible, I would ask you to think of the colour spectrum. It contains bands but no boundaries. The different colours stand out clearly, as distinct and exclusive, yet close inspection shows that there is no boundary between them. The spectrum provides a paradigm of classification, yet it is boundaryless.

He continues ([2], p. 182)

... We must shift away from the classical perspective. We are carried away by images which make us find boundarylessness problematic. We think of a system of classification as like a grid, a system of pigeon-holes, a way of drawing a line, dividing a field. In this way of thinking, Frege's idea that a boundaryless concept is no concept at all seems irresistible. But we should shift images. Classification is better likened to providing magnetic poles around which some objects cluster more or less closely and from which others are more or less repelled; some fall between a number of poles, drawn by more than one but especially close to none.

I have quoted Sainsbury at length here in order to emphasize that the criticisms I have been making focus on aspects of his discussion which are in any case rather distant from his main concern. This is not the occasion to attempt to appraise the conception of vagueness which the passage conveys, but I certainly don't venture to deny that there is something correct in the adjustment to much contemporary thought about vagueness which Sainsbury is trying to teach. I merely offer one deflationary thought.

It is one question whether the idea of boundarylessness offers a useful fresh perspective on the nature of vagueness. It is a different question whether the adjustment points to an understanding of vagueness which is paradox-free. Even if the 'classical conception' mislocates what defines vagueness, it is quite another matter whether it altogether misdescribes it. Sainsbury himself acknowledges that 'a boundaryless predicate allows for borderline cases' - so vagueness is associated with a tripartite division, or taxonomy of relevant cases, even if this is 'not its defining feature'. And now, why should it matter whether a feature is a defining feature or not, provided it is a feature? How could the classical conception have been led to avoidable paradox by correct characterization of non-defining features, even if it mistakenly took them to be defining ones? Above all, how is the conception of vagueness as boundarylessness fundamentally at odds with the characteristic sentence approach — why should there not be 'a sentence schema, containing a schematic predicate position, such that the sentence resulting by replacing the schematic element by a predicate is true iff that substitute is a boundaryless predicate? And why in particular is the approach illustrated by (vi) not suitable to generate such a sentence-schema? Perhaps these questions somehow miss the whole point about boundaryless predicates. If so, it will be interesting to learn how.

I tentatively conclude that the case — which *must*, of course, be flawed! — for thinking that higher-order vagueness is *per se* paradoxical is so far unanswered.

The University, St Andrews, Fife KY16 9AL

University of Michigan, Ann Arbor, MI 48109, USA

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